GRAPHS AND STATISTICS

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GRAPHS AND STATISTICS

A SUGGESTION

FOR

A FINISHING COURSE IN MATHEMATICS

BY

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With Two Art Plates

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1926

" He that seeketh findeth "—he finds what he seeks, as we know from the use of quotations and statistics.

T. R. GLOVER.

Mathematics may be compared to a mill of exquisite workmanship which grinds you stuff of any degree of fineness : but, nevertheless, what you get out depends on what you put in; and as the grandest mill in the world will not extract wheat-flour from peas-cods, so pages of formulæ will not get a definite result out of loose data.

T. H. HUXLEY.

Mathematics is that study which knows nothing of observation, nothing of induction, nothing of experiment, nothing of causation.

T. H. HUXLEY.

Every mathematical book that is worth anything must be read " backwards and forwards "..... ...the advice of a great French mathematician, " Allez en avant, et la foi vous viendra."

G. CHRYSTAL.

PREFACE

A title adequate to express the scope and aim of this book has not been found. To the word "Graphs" a wider meaning than is customary has had to be given—a meaning which is not unfortunate, for special types of graphs have already their own names; and "Statistics" refers to no more than a sketch of the nature of that science. This book is no treatise. In its present form it is not even a textbook. First and last, it is meant to be "suggestive", not "impressive".

The genesis of the book was a sense of unfair treatment meted out to the majority of First Year college students in Bombay in their study of geometry and algebra. In common with many students elsewhere they have to submit to a discipline which is meant primarily for the small minority who are to make Mathematics their special study. The best was made of this irksome situation, but the lifeless and deadening memory-discipline which it meant for most of the students could not be regarded with equanimity.[1] To suggest ways in which the last impressions of mathematics, received by students who are to devote themselves to other studies, may be made more vivid and friendly was the first purpose of this book.

But, as the writing of the chapters proceeded, there was opened out the possibility of doing some service to those who have to face quantitative problems in the non-

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1 In Bombay the situation has unexpectedly become less difficult than it was when the writing of this book was begun, for it has been decided to make mathematics an optional subject among First Year studies. Few teachers of mathematics will regret this. Yet it may be that in the long run the effect of the change will be negligible: for recognition of the essential worth of some training in mathematics may lead to a refusal of valuable opportunities to those who have not submitted themselves to a discipline of this type.

mathematical sciences. This was especially so with regard to physiology, and much was added simply because of the applications of mathematical devices that had actually been made by physiologists. The medical bias that exists is not intentional. It may turn out to be inevitable; for the subjects to which applications have been made, have a more direct interest than those we have been accustomed to think of as illustrative of mathematical methods. Further, the difficulties of defining the quantities investigated are much less even in metabolism than in engineering, or in actuarial science,[1] or in psychology. It may be, however, that illustrations such as would be suitable here, will be found in abundance in psychology or in agriculture.

To give every opportunity to test the comprehensiveness of what is here submitted full references have been given to the authorities that were available. The nature of many of these references will make clear that this book is not primarily a textbook. Also it is not self-contained; where material is easily available elsewhere, the treatment has been made as brief as possible. In Mathematics conciseness is specially a virtue, but it may easily become a difficulty for the student of the type for whom this book is ultimately meant. However, it will probably be found, when once the novelty of the subject-matter has passed, that nothing is given which is essentially more difficult than what is found in the mathematical textbooks customarily used—certainly not more difficult than some of the jokes and allusions that occur in the books read at this stage in Literature! With the help of a teacher any student should be able to benefit from what is written here. But the question as to difficulty need not be kept in the forefront now: obviously the treatment can be greatly extended or curtailed, as experiment may show to be advisable. No attempt has been made to produce examples merely for practice. Just what it is feasible to teach can be discovered only through

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1 For applications to these v. Lipka's "Graphical and Mechanical Computation" and Elderton's "Frequency Curves and Correlation".

experiment, and that only after the novelty of the dress given to the mathematical methods has ceased to be felt by both teachers and pupils: the effect of direct interest cannot be judged beforehand.

Many mathematicians will probably be disappointed in finding so little mathematics in these proposals. It would be very easy to strip the mathematical methods of their dress and present them concisely in rigid form: but it will be frankly recognised that it is impossible to teach Mathematics to the students who are contemplated here. Yet they do need to be given an opportunity to appreciate in some measure how mathematical methods can be applied. A more perplexing question is whether a course of this kind will adversely affect the skill of those who are going to specialise in mathematics. The breadth of outlook, imparted both in the applications and in the allusions to the nature, relations, and limitations of mathematics, will be a distinct advantage. It is generally true, moreover, that those who are fit to specialise in mathematics feel the direct attraction of the transformations of symbols, skill in which is essential to their progress in mathematical studies. It would probably be no hardship if aspirants to special mathematical knowledge were tested by the colleges at the beginning of their course as to their skill in these essential manipulative processes.

But the general impression with regard to the material presented will be one of amusement at the idea that the processes described here are elementary. In both economics and medicine things are dealt with which must, as things are now, be excluded from the degree courses. Yet, when the methods discussed are examined, it will probably be found that the only idea which has not been admitted to be elementary is that of functional scales. The treatment of the calculus is very limited and frankly imperfect ; no more than mention is made of e, either in connection with the calculus or with the normal curve of error (for e suggests a beginning rather than a climax) ; the introduction of many coordinates involves no difficult extension of the ideas for two coordinates. Even functional scales have not been

treated in so formal a way as to make them forbidding. They might have been treated more systematically, but the method of dealing with them in particular cases, as they actually occur has been preferred. How best to present this idea is a question that can be investigated: it is certainly a tremendous advantage to be able to regard functions in such a concrete and eminently useable way.

Dissatisfaction way well be felt with the vagueness of what is presented in the latter part of Chapter VI. This is the part of the book that is most difficult to treat satisfactorily, and where there is most danger of going to an extreme in the undogmatic attitude generally adopted here. It has to be remembered, however, that our purpose is even less to teach statistics than to teach mathematics; a real understanding of the former involves a knowledge of much more of the latter than we can contemplate in these pages. All that has been attempted is to put the student in a position to appreciate some of the main possibilities and limitations of statistical methods. Though the ultimate test is in the intelligibility of results, there is a danger of making things easier than they are by merely presenting formulæ in which quantities are to be substituted without consideration of much more than the superficial meaning of the symbols; confidence here can easily become unscientific. Emphasis has been laid rather on the thorough examination of a limited number of examples: it will be of interest to discover how far this method can be made to appeal to students. It involves sustained effort, but much can be accomplished through emulation stirred up by dividing the work among sets of students. It makes a world of difference in interest, as well as in effects, if a set of figures is actually handled, and not merely regarded in a superficial way. Still more enlivening would be the effect, if students could be induced to combine in preparing their own data—when the almond tree sheds its leaves (**7.22** Ex. 1)!

Explicit reference should be made to the constant effort to suggest the possibility of generalisation. One of the most important intellectual habits to form is that of extending one's thoughts in a systematic way. The converse aspect

of this is the stress that has been laid on the interrelatedness of apparently very different things. This gives an unexpected unity to a book which professedly proceeds by the selection of what is elementary.

Attention may be called here to some matters of detail. How this book is related to the subjects more usually taught may be seen from the three pages of *notanda* at the beginning. There is no need to treat the definitions and fundamental propositions there referred to otherwise than briefly. Trigonometry as taught at present must be retained, for direct applications of it are very frequent The knowledge of graphs with which many students are equipped is inadequate, but it can easily be supplemented.

Difficulties will be encountered in examining large classes in the practical aspects of what is described here; but these can be overcome. In Wilson College it has been customary to provide candidates with numbered copies of tables of logarithms, etc.; this makes it possible to trace any irregularity with greater ease. These books are collected ten minutes before the examination closes, and so confusion is avoided. The manner of conducting short slide rule examinations is fully explained in connection with **7.31** Ex. 3.

A book like this which breaks new ground, both in the aim with which it is written, and in the type of subject-matter it presents with that aim in view, must necessarily contain many faults. These would have been even more patent had it not been for the generous help I have received from Mr. K. M. Kharegat, of the Tata Engineering Company, who made nomograms more of a reality to me and gave ready aid in other ways. The help received from Professor V. M. Khanolkar of the Seth Gordhandas Sunderdas Medical College, Lt.-Col. J. Morison and Major Sokhey of the Haffkine Institute, and Dr. Muir of the Calcutta School of Tropical Medicine and Hygiene, has been invaluable. Professor K. R. Gunjikar of Elphinstone College has kindly read the early proofs of the first eight chapters, and made several salutary suggestions. To all these I am grateful for the help they have given. In no sense can they be responsible for the imperfections that remain; for to no one has it been possible to show the book as a whole. The necessity of leaving Bombay on furlough has made the final stages of the preparation of the book rather hurried. To Mr. B. D. Mahatme of Wilson College, I am also indebted for revision of the page-proofs, and to many friends who have helped by placing material freely at my disposal.

Perfection in a book which ranges so widely can be approached only by the cooperation of many, and I shall be grateful to receive criticism of any kind, especially such as may make it possible to avoid what is merely scrappy and inept, and make it unnecessary for students in their future studies to unlearn anything in phraseology or fact which is presented to them here.

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October, 1926.

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TO THE STUDENT

No book on mathematics is written so as to leave you nothing to do save think; the best thought comes after action. This book calls you specially to action. Many of the diagrams have been drawn without a ruler so as to remind you that you must test mathematical statements by sketches if you are to learn, and that you must depend on hand and eye to represent rapidly for you the essentials of a graphical relation,—even when scales enter into the figures. You may miss almost as much through being over-exact as through being untidy; you must learn to estimate as well as to prove. The diagrams are also frequently grouped in a way that appears to be inconvenient; but this should stimulate you to multiply the figures, either in the margin of your book or elsewhere: none of the figures are meant to be authoritative. (Note, however, how figures have been placed on the pages so as to make comparisons as easy as possible.) Again you are called to look up other books, not merely to trust this book: many of the references you will recognise as meant for doctors and others who have to be critical of the non-mathematical statements made here; but most of them relate to books which are in every college library. You will find it a real pleasure to range through these books. Details in them are criticised when occasion arises; your activity should extend to thoughtful criticism of what is written here also.

Your criticism, however, should usually be restricted to the mathematical methods. As in all serious study, you are asked in this book to work with ideas (in medicine and in money) which you do not fully understand. You must not think that, having used these ideas, you have comprehended them. Before you can appreciate fully what is written here, about engines or inheritance or eating, you have to study in preparation many fundamental ideas. For instance, with regard to metabolism you will learn that

proteins are not all of one kind, that vitamines have to be considered quantitatively, that mere bulk is an important consideration in choosing food. So too the steam which passes through an engine has in our treatment been all too simplified. Dr. Muir and others tell me that the representation of leprosy by curves is so much a secondary matter that it might be made a misrepresentation of this subtle disease. You have great reaches of chemistry and physics and physiology to traverse before you can appreciate truly the ideas we have simplified here.

In this book you see Mathematics as a servant, a humble servant, of the other sciences. You get only glimpses of her queenly glory.[1] She knows there are regions she can as yet only dimly illumine—some in which she may not aspire to shine at all. But she will be well content if she can pass on to you something of her methods of orderliness and elasticity, consistency and thoroughness, ease and brevity of representation—there *is* a mystery in symbol rationally used.

You will have a great deal to do with scales in this book. If you are to keep essentials before you, you must respect these scales. Do not mark on a scale of reference more than the fewest graduations that will enable you to read with accuracy the position of a point: selected graduations should be marked with either even or decimal numbers. NEVER mark on a scale the graduations of the given points. Otherwise you show that you are crowding your mind with detail, or that your attention is wrongly directed. (Even in figure 46 no marks are placed on the scales of reference themselves.) The importance of the given values should be noted in some other way than on the scales. When drawing a graph too, it is a good thing to mark no points on the line save those given. Readings obtained, say, by stretching a thread over the diagram, should be put down in their proper place in your scheme of work.

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1 "Gauss considered mathematics to be 'the Queen of the Sciences, and arithmetic [not the arithmetic *you* know] the Queen of Mathematics'".

You should ever be devising arrangements which are clear and concise. The conciseness of many of the tables in this book may sometimes be puzzling, but the effort spent in mastering and criticising them will be well repaid. In these tables much is made to depend merely on the relative positions of numbers, and much verbal explanation is dispensed with. (The table on page 141 is an example: in it there are really two tables, one to the left, not completed, and another to the right: in the first and third columns "log" is understood to be repeated all the way down.) In general, repetition of any kind is avoided, e.g., in giving a series of dates, or in stating intervals, or in writing determimants or other expressions which are obtained by cyclic substitution. Conciseness is illustrated too by the use of as many intelligible contractions as possible. Standard abbreviations, like *v.*, *i.e.*, should be read as part of the sentence: other contractions suggest their meaning—1·66+ indicates that the value is "1·66 or a little more"; a numerical index after the title of a book refers to the edition which is quoted; in **9.32** it is considered unnecessary to explain that the constant 10^{c_1} is replaced by c, etc. One aim of this book is to make you quick to realise the possibility of such simplifications of detail: if so we be set free, we may learn to appreciate "means we could not smell as flowers towards ends we could not taste as fruit".

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to note carefully the numbering of paragraphs in a chapter: it shows how the facts should be arranged in your own mind and you should try to fit new facts into their proper place in the scheme. In some places other systems of numbering may be possible, and you may prefer one of them; you should not hesitate to make and use your own system,—and to discard it if it turns out to be defective!

Below are noted important points you have already studied:

I. **Cross Multiplication.** To solve the simultaneous equations

$$\left.\begin{aligned} Ax+By+C=0 \\ ax+by+c=0 \end{aligned}\right\}$$

think of the coefficients as arranged thus:

$$\begin{array}{ccccc} A & B & C & A & B \\ a & b & c & a & b \end{array}$$

and write $$\frac{x}{Bc-bC}=\frac{y}{Ca-cA}=\frac{1}{Ab-aB}.$$

With practice all this can be done **mentally**; thus to solve $\begin{aligned} 6x-5y+2=0 \\ 4x+3y-7=0 \end{aligned}$ write at once, or think of, $\frac{x}{29}=\frac{y}{50}=\frac{1}{38}$.

This is really a simple case of **1.83.**

II. A straight line may be fixed by

(i) the **intercepts** a, b it makes on the axes; its equation is then $\frac{x}{a}+\frac{y}{b}=1$; (generalised in **5.4.**)

(ii) the **slope** m and the *point* $(0, c)$ in which it cuts the y axis: in terms of these the equation is $y=mx+c$.

(The slope is the ratio, difference of ordinates : difference of abscissae.)

III. **Counting Squares** on graph paper for the area of a curve: along the boundary of the curve the squares are cut into by the curve: unless there is an obvious excess of partial squares with more, or with less, than half their areas included in the curve, the most workable rule is to count the total number of partial squares and divide their number by two.

IV. The **Quadratic Graph** $y=ax^2+bx+c$ is a parabola with its axis parallel to the y axis, and concave upwards if $a>0$, downwards if $a<0$.

V. $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, if $ax^2+bx+c=0$. This is a very excellent mantra: it makes many things clear (**1.21, 4.2.**) b^2-4ac is called the **discriminant** of the expression ax^2+bx+c.

VI. Before reading this book you should know

(i) how to use tables of logarithms, etc.;

(ii) that equiangular triangles have their corresponding sides in the same ratio and their areas in the ratio of the squares of corresponding sides.

(iii) that the sum of any number of the decreasing terms of the series a, ar, ar^2, ar^3, ar^4,..... ar^{n-1}, ar^n, ar^{n+1},..... where $r<1$, is never greater than $a/(1-r)$, and can be made as close to this quantity as we like by taking n large enough;

(iv) that the number of ways in which n things can be combined r at a time is $\frac{n!}{r!\,(n-r)!}={}_nC_r$;

(v) that the probability of an event which can happen in p ways and fail in q ways is $\frac{p}{p+q}$; and that if another independent event can happen in p_1 ways and fail in q_1 ways, the probability of both happening together is $\frac{pp_1}{(p+q)\,(p_1+q_1)}$; and

(vi) that work done is measured by the product of the force with which it is done and the distance through which it is done.

VII. On all possible occasions use detached coefficients, and complementary division (the "Italian" method): the latter method is specially useful also for finding how much less the sum of several numbers is than a given number.

CHAPTER I

MISCELLANEA

(It has not been found possible to fit the subjects dealt with in this chapter into the general plan of this book, and the student, unless he have a bent towards mathematics, is advised to refer to it only as he finds need. Chapter II is the best starting point.)

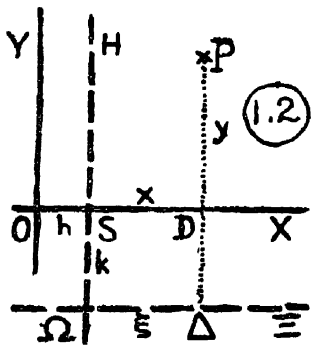

Fig. 3

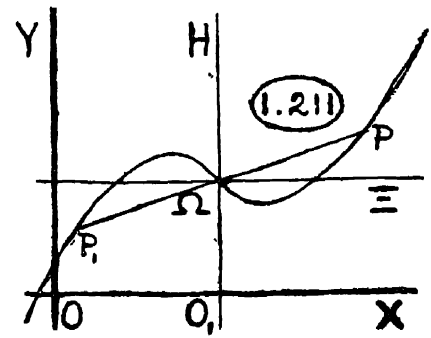

Fig. 4

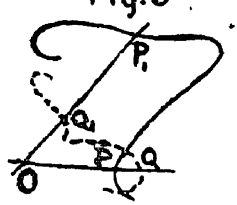

Fig. 5
Central Inversion

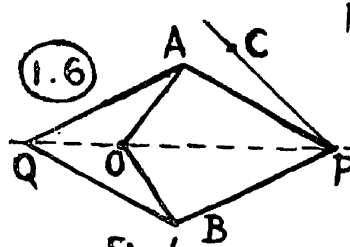

Fig. 6
Peaucellier's Cell

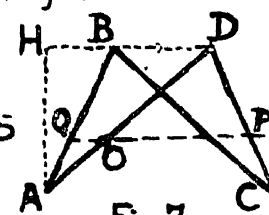

Fig. 7
Hart's Linkage

1.1. To reverse the process of addition or subtraction of algebraic fractions is often difficult; but a very simple rule enables us to find the numerator of a **partial fraction** as it is called, in which the denominator is a linear factor of the denominator of the given fraction. The rule is as follows:

$$\text{if } \frac{N}{ax^n+bx^{n-1}+\cdots+k} = \frac{A}{x-r} + \frac{B}{ax^{n-1}+b'x^{n-2}+\cdots+k'}$$

where N is an expression of lower degree than the denominator, then A is found by substituting r for x in the given fraction, with the factor $x - r$ omitted.

The truth of this is easily seen by multiplying through by the denominator of the given fraction, and then putting x equal to r. The term containing B then vanishes.

Test this in particular cases, *e.g.*, $\frac{6x^2-8x+4}{(4x^2-3x+1)(x-1)}$; such examples students can prepare for one another by adding together suitable fractions. Those who are specially interested will find the matter treated at length in almost any textbook on algebra.

1.2. PARALLEL DISPLACEMENTS OF AXES. A curve in a plane is not affected by a change of the axes with reference to which the coordinates on it are measured. If the axes change their position, a corresponding change takes place in the coordinates, and the *equation* which represents the curve alters its form. The position of the axes may be chosen so as to make this equation as simple, or as convenient, as possible.

The simplest change of axes is that in which the origin is moved to a point (h, k) and the axes retain their directions. This is represented in figure 3, where k is negative: in all cases we have for the coordinates of any point on the curve

$$x = OD = OS + SD = h + \xi$$
$$y = DP = D\Delta + \Delta P = k + \eta,$$

where ξ and η are the coordinates of a point P with respect to the new axes. (If you have difficulty with the negative value of k, redraw the figure with Ω somewhere in the first quadrant, and you will at once see that these statements hold true.) When we substitute $\xi + h$ for x and $\eta + k$ for y, we get a new equation in the variables ξ and η: this equation may or may not be simpler than the original, but it represents the same curve.

1.21. If in the above we suppose $k = 0$, the displacement of the origin is simply along the x axis. When this transformation is applied to $y = ax^2 + 2bx + c$ (p. 3, IV) so that $h = -b/a$, we have

$$y = a\left(\xi - \frac{b}{a}\right)^2 + 2b\left(\xi - \frac{b}{a}\right) + c = a\xi^2 - \frac{b^2}{a} + c = a\xi^2 - \frac{b^2 - ac}{a}.$$

Geometrically this corresponds to placing the curve so that it is symmetrical (**2.12**) with respect to the y axis: it cuts the y axis at a distance $-(b^2-ac)/a$ from the origin, whether b^2 is greater than ac or not. You can easily with the help of a figure connect this fact with the condition for real roots. (cf. V)

1.211. SOLUTION OF CUBICS. Similarly, by putting $x=\xi-b/a$ in $y=ax^3+3bx^2+3cx+d$, the equation can be transformed so that there is no term in x^2; then

$$y=a\left(\xi-\frac{b}{a}\right)^3+3b\left(\xi-\frac{b}{a}\right)^2+3c\left(\xi-\frac{b}{a}\right)+d$$

$$=a\xi^3+\xi\left(3\frac{b^2}{a}-6\frac{b^2}{a}+3c\right)-\frac{b^3}{a^2}+\frac{3b^3}{a^2}-\frac{3bc}{a}+d$$

Writing this $\eta\left(=y-\dfrac{2b^3-3abc+a^2d}{a^2}\right)=a\xi^3-\dfrac{3}{a}(b^2-ac)\xi$,

we see, since the terms in ξ and η are of odd degree (**2.12**) that the curve is symmetrical about the origin for ξ, η, *i.e.*, about the point in which the curve cuts the second y axis. (Fig. 4.)

Ex. Apply Descartes' Rules (**2.17**) to explain the significance of the sign of the coefficient of ξ in the last form of the equation to the cubic. Draw figures for the different cases: use particular values of a, b, c to make the curves definite.

The possibility of **removing the second term** from a cubic expression gives an easy way of getting a graphical solution for any equation of the third degree. Let the equation, simplified by the removal of this term and by division by the coefficient of x^3, be

$$x^3-px-q=0.$$

This may be regarded as $x^3=px+q$, and so, if on a **carefully drawn graph of $y=x^3$**, the straight line $y=px+q$ is drawn, the points where the ordinates of these graphs are equal, *i.e.*, the points of intersection, give the values of x, possibly three, for which $x^3-px-q=0$. From these values have to be subtracted the b/a referred to above in order to get the roots of the original equation; for the equation we are here considering is, in the earlier notation, $\xi^3-p\xi-q=0$, and we wish to find x, which is $\xi-b/a$. (Cf. **2.16**, Ex. 3; also, say, Davison's "Higher Algebra", p. 190.)

Ex. 1. Solve $2x^3-5x+2=0$. Construct such examples for one another; draw the corresponding figures. (Cf., *e.g.*, Barnard and Child, "A New Algebra," p. 418, Ex. 11) (In using a reference graph of $y=x^3$ to get approximate values of the real roots of $x^3-px-q=0$ it is easiest to **stretch a fine thread across the graph** in the position $y=px+q$: cf. p. 2, II ii and **1.33**).

Ex. 2. Apply the transformation of **1.2** to curves like those in **2.1**, *e.g.*, move the origin for $y=x^2$ to (2, −3), that for $xy=3$ to (3, 1).

Ex. 3. Move the axes in **2.0** Ex. 3 so that the hyperbola there obtained is expressed by the simplest equation possible.

1.31. ACCURACY IN READING SCALES. It is important to be able to use our tools so as to give us results which are the best possible. With an ordinary footrule we should be able to measure habitually to 0·01 of an inch. So also with a good scale of centimetres. If a length comes between two successive tenths you must estimate the number of **hundredths** of a centimetre extra which should be included in the measurement. If you find difficulty in making such estimates, you should mark a scale of inches or centimetres on a line, get a friend to put points on this line in any positions, write down your estimates of the number of tenths these points are from a graduation, and check your work by actual measurement with a rule. A little practice of this sort will make you confident in the finer measurement that is required in graphical work.

1.32. DRAWING A TANGENT TO ANY CURVE at any given point on it. For a circle the perpendicular to the radius gives us the tangent. But for most other curves we have only the definition of the tangent as *the limiting position of a chord through points on the curve near the given point* to work with (cf. **3.12**). In practice we cannot of course go to the limit: but satisfactory results may be got by devices such as the following :—

(i) Measure off equal lengths *along the curve* in opposite directions from the point, and draw through the point a line parallel to the chord joining the ends of the equal arcs.

(ii) It is usually easier, and quite as satisfactory, to take, instead of the above chord, that joining the points the abscissae of which have the abscissa of the given point

as their arithmetic mean, and draw the parallel to this chord; thus for $y=x^2$ we take as the slope of the tangent at any point $\{(x+h)^2-(x-h)^2\}/2h$, or $2x$: cf. Barnard and Child, "A New Algebra", p. 606. For other curves than the parabola the method is not exact; but with careful drawing it is a good one. (**3.121, 3.13.**)

Ex. With a horizontal scale of $1''=0{\cdot}5$ and a vertical scale of $1''={\cdot}05$ plot carefully from a table of logarithms the graph of $y=\log x$ between $x=3$ and $x=5$. Draw the tangent at $(4, \log 4)$ and measure its gradient. Check your result by formula (iii) of **3.13.**

1.33. DISCOVERING LAWS. In **1.31** we have noted how accurate work in measurement depends on estimating the position of a point relative to two other points in the same line, the graduations on either side of it; and how accuracy is attained only by practice. Very often it is important to be able to estimate the position of the straight line which is closest to a number of points which lie more or less along a straight line: the ability to do *this* depends very much on practice.

Such assemblages of points occur frequently when the results of experiments are plotted with reference to suitable axes: cf. **9.3, 9.4.** A very simple example is given in **6.22,** the determination of the value of π by measuring circles of different sizes. But the method is not restricted to the case of lines passing through the origin; whatever the trend of the points, it is possible to move a fine thread among them until it appears that it *passes evenly through them* and *as close to them as possible*—the points on one side of the stretched thread are, on the whole, as near to it as those on the other. In drawing a tangent it was found possible to avoid depending merely on an estimate by eye; but here no other procedure is possible; yet this method is the graphical equivalent of a very elaborate arithmetical method to which you will often see reference—the *method of least squares*, the squares being squares of the distances, in some direction, of the points from the line; these distances are squared in order to get rid of the cancelling effect of opposite signs (**1.5**). Once the line has been drawn its equation can be read off by the rules of p. 2, II: great care must be taken to choose suitable units and to manipulate them correctly.

It is necessary here to do little more than call attention to this method. Those who need help should practise it with the aid of a book which deals with difficulties in detail. Collections of examples may be found in many such books, *e.g.*, Barnard and Child, "A New Algebra", p. 500; Usherwood and Trimble: "Intermediate Mathematics", p. 118; "Practical Mathematics", I 226, II 122.

A few examples are added from "Practical Mathematics",[1] by Professor Perry, who expresses himself very vigorously (p. 64) in favour of this as against the algebraic method because of the type of uncertainty mentioned in the note in **6.3**, — an uncertainty that can be detected easily on a drawing.

Ex. 1. The points (2, 5·6), (3, 6·85), (4·5, 9·27), (6, 11·65), (7, 12·75) (9, 16·32), (12, 20·25), (13, 22·33) lie most nearly on $y = 2{\cdot}5 + 1{\cdot}5\,x$.

Ex. 2. When the weight A was being lifted by a laboratory crane, the handle-effort B (the force applied at right angles to the handle) was measured and found to have the following values:

A	0	50	100	150	200	250	300	350	400
B	6·2	7·4	8·3	9·5	10·3	11·6	12·4	13·6	14·5.

Show that the law for A and B is $B = 0{\cdot}207A + 6{\cdot}3$.

Ex. 3. Show that the following observed values of x and y obey the law $y = 1{\cdot}8 + 0{\cdot}1x^2$: (1·1, 1·91), (1·8, 2·13), (2·5, 2·42), (2·9, 2·65), (3·6, 3·09), (4·3, 3·66), (4·8, 4·09), (5·4, 4·73).

(A rough plotting of these points should suggest a curve of the form $y = x^2$; plot then carefully corresponding values of x^2 and y.)

Ex. 4. Find the law connecting P and W in the following table:

Indicated Horse Power, P	36·8	31·5	26·3	21	15·8	12·6	8·4
Steam used per hr. per *I. H. P.*, w.	12·5	12·9	13·1	13·3	14·1	14·5	16·3.

(The law in this case was first discovered as $W = 37{\cdot}5 + 11{\cdot}5\,P$, where W is the *whole* weight of steam used per hour, *i.e.*, wP.: the law may also be written $w = 37{\cdot}5/P + 11{\cdot}5$, and so may be discovered by plotting w against $1/P$. Try it both ways. Cf. also Lipka's "Computation", p. 135.)

Ex. 5. Find the law connecting x and y for the following values:

x	0	·05	·1	·3	1	1·4	2	2·5
y	0	·136	·26	·55	·97	1·1	1·22	1·24.

(Here at first apparently $y \propto x$, but later it seems to be approaching a limiting value. This suggests $y = ax/(1 + bx)$, *i.e.* $y/x + by = a$. Accordingly plot y/x against y. The law found is $y = 3x/(1 + 2x)$.)

Perry suggests rules to be observed when, as often

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1 Six lectures delivered in 1899: published by the Board of Education, 1907; pp. 183. Price Ninepence.

happens, there is an indication of the *rate* at which y changes with respect to x:

if $Dy \propto y$, $y = ae^{bx}$ **(3.22** ii, **9.3)**; plot $\log y$ against x:

if $Dy \propto y/x$, $y = ax^b$ **(3.22** i, **3.211, 9.4)**; plot $\log y$ against $\log x$:

if $Dy \propto x$, $y = a + bx^2$; **(3.22** i **)** plot y against x^2.

1.34. POSSIBLE ERRORS. No measurement is absolutely accurate: all that can be said is that the quantity measured lies between certain *limits*, *upper* and *lower*. Thus, if we say that a certain length is $1{\cdot}7''$, we mean that it is some length between $1{\cdot}65''$ and $1{\cdot}75''$; and so on. It is often necessary to know how this fact affects the result of calculation when measurements are combined—what is the possible error in the value deduced? This matter is treated with thoroughness and generality by Barnard and Child in "A New Algebra", Chap. XXVIII, and here we note only the particular result that the possible error of a sum or difference of two similarly measured quantities is double that of the original measurements, *i.e.*, the error in the difference between measured lengths, $2{\cdot}3$ cms. and $1{\cdot}7$ cms. may be as much as $0{\cdot}1$ cm.[1]

Ex. 1. Show that in **3.151** Ex. 2 the value of the separation deduced by different experimenters between two wave numbers corresponding to the wave-lengths there specified may differ by 20 units. (One might say 22 units, but that is being over-precise in speaking about *possibilities*. This example is taken from Phil. Trans. R. S. *A*, **225** 361.)

Ex. 2. The example of error in area given by Barnard and Child, p. 321, may be expressed more conveniently thus:

in $(1{\cdot}8 \pm {\cdot}05)(0{\cdot}7 \pm {\cdot}05)$ the greatest possible departure from $1{\cdot}8 \times 0{\cdot}7$ is $\cdot05\,(1{\cdot}8 + 0{\cdot}7) + (\cdot05)^2$ *i.e.*, $\cdot125 + \cdot0025$ or $\cdot1275$. (Cf. **3.151.** Ex. 1.)

Ex. 3. Represent graphically as in **7.531**, or otherwise, the general rules given by Barnard and Child, *op.cit.*, p. 320; select cases in which A and B have varied relative values and signs:

If a, a' b, b' are given numbers and A, B are any numbers such that

$$a < A < a' \quad \text{and} \quad b < B < b',$$

then (i) $a + b < A + B < a' + b'$;

(ii) $a - b' < A - B < a' - b$;

and, if all the letters denote positive numbers,

(iii) $ab < AB < a'b'$;

(iv) $a/b' < A/B < a'/b$.

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1 Another treatment of this subject which some may prefer will be found in King's "Statistical Method", Sec. 44.

1.341. An attitude, the converse of this, occurs when for simplicity we discard differences from the true value of a quantity as negligible and cumbersome. A simple example of this type of **approximation** to a value is when we seek the reciprocal of, say, 1·002. By division we know that $1/(1+\alpha)=1-\alpha+\alpha^2-\alpha^3+\ldots\ldots$; when α is small, we can consider α^2 and higher powers of α negligible. And so in the particular case mentioned the required reciprocal is $1-\cdot002=\cdot998$. Compare this with the values got by taking successively more terms in the expansion—0·998004, 0·998003992, etc. This process is closely connected with that in **3.121.**

1.4. PROPORTIONAL PARTS OR INTERPOLATION: When values of a quantity, *e.g.*, logarithms, are given in a table, it is impossible to give all values; they have to be given at close, regular intervals, and, if they are required at intermediate values, it is assumed that the quantities change regularly in the interval[1]. Thus in the region 41 to 45 of a table of logarithms between any two successive tabular values, say, those for 434 and 435 the logarithm increases steadily by units for each figure in the fourth place from 6375 to 6385; so also in the region 210 to 219, where the rate of increase is 2 units.

1.41. But the rate of increase is not usually so regular. Plot carefully on a large scale the nine logarithms between the numbers 2·63 and 2·64; when you stretch a fine dark thread through the first and last of these you will find that five points lie above the line while only four lie below it, and these are much closer to it. This indicates that the true curve of logarithms lies rather above the straight line, and this agrees with the general nature of the graph for $y=\log x$—it is concave on the lower side and to the right: Cf. **1.32** *Ex.*

This test is not quite decisive, for we do not know that the end points are the true positions of $(x, \log x)$ — the values are given only to the nearest integer

1 For interpolation with another assumption *v.* Horst von Sanden, "Practical Mathematical Analysis" (Metheun) p. 64. Cf. **1.32** (ii) and the remark in **2.31** about a curve through n points.

in the fourth place; but it is quite a good test. Note how in the instance given the points group themselves in twos and threes successively in straight lines: this is because values have to be given to the nearest integer.

1.42. In some places the rule of Proportional Parts is inadequate because the changes in the function are so slow, *e. g.*, in the upper parts of the table for log sin x°; if the successive values of the function are to be distinguished, an extra number of decimal places must be taken: but there is usually no advantage in this. At the lower end of the table of logarithmic sines the changes in the function are so rapid that proportional parts cannot be used till the differences begin to be regular, and so in Knott's Tables a special page is given to the values of log sin x for values of x up to $7^\circ\ 50'$. So also for the lower part of the table of logarithms, which in Chambers' Mathematical Tables is given to an extra decimal place.

An interesting comparison, which illustrates a difficulty in compiling really good tables, may be made from Knott's Tables (1905) pp. 2, 3, on both of which logs are given of numbers up to 1109.

Nos.	p. 2	p. 3	Nos.	p. 2	p. 3	Nos.	p. 2	p. 3
1010	0043	3	1050	0212	2	1080	0034	4
1	7+	8	1	6	6	1	8	8
2	51+	2	2	20	0	2	42	2
3	5+	6	3	4	4	3	6	6
4	60+	0	4	9−	8	4	51−	0
5	64+	5	5	33	3	5	5−	4
6	8+	9	6	7	7	6	9−	8
7	72+	3	7	41	1	7	63−	2
8	6+	7	8	5	5	8	7−	6
9	80+	2	9	9	9	9	71−	0

The logarithms on page 3 are the more accurate; it is interesting to test them by comparison with still more accurate tables, *e.g.*, Chambers'. The nature of corrections that are required in the logarithms of page 2 is indicated above by signs, and it is seen that these are in opposite directions at the two ends of the range to which the mean differences selected apply, while in the middle of the range the logarithms have almost no errors. It would seem better to have omitted the differences on page 2, as is done on p. 8

but the adjective in the title "Mean Difference" emphasises the limitation of the results to be expected from these tables.

Ex. 1. Make a comparison like the above between the portions of Knott's Tables on page 8 or page 12 which overlap with page 7.

Ex. 2. Investigate the effectiveness of the device in Castle's Tables—separate differences for the first and second halves of the line—to overcome the above difficulty.

1.43. LOGARITHMIC INTERPOLATION: When the values of variables are given not directly, but as functions of the variable, the process of interpolation has to be modified: but the same linear assumption may be made. Tables for vapour pressure of water at different temperatures depend upon a formula (**5.3** Ex. 2) which gives $\log p$ as a function of the temperature. In Kaye and Laby's "Physical and Chemical Constants" where several such tables are given (pp. 40-43), it is shown how intermediate values of the vapour pressure should be calculated. Values of vapour pressure given in different tables[1] do not quite agree: take 9·14 and 17·36 as the values at 10° and 20°. To find the value for 16° we have to seek the number for which the logarithm is

$$0{\cdot}4 \log 9{\cdot}14 + 0{\cdot}6 \log 17{\cdot}36.$$

This expression can with little difficulty be found to be $\log 13{\cdot}43$ and so the vapour pressure at 16 is 13·43—not

$$0{\cdot}4 \times 9{\cdot}14 + 0{\cdot}6 \times 17{\cdot}36 = 14{\cdot}07.$$

Ex. 1. Test one of the tables of vapour pressure of water to see whether the values between two chosen values actually agree with this formula.

Ex. 2. If $y = 10^x$, find the values of y corresponding to x_1, $1\frac{1}{3}x_1$, $1\frac{2}{3}x_1$, $2x_1$

This is the problem of finding, say, 9 geometric means, **6.141**, between two quantities: cf. Whipple's "Vital Statistics", p. 143. In **2.21**, $P_n = P_0(1+r)^n$, it means finding what P_0 amounts to in time n. Geometrically this is equivalent to considering a straight line on semi-logarithmic ruling (**9.3**).

As intermediate points on a straight line can be found, so points on the line produced can be definitely located: an example of this is given in **9.3** Ex 6, but the process of **extrapolation** is difficult and risky because of the possibility of conditions changing (cf. also **6.3** Note).

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1 Gregory and Hadley's Physics, p. 560, Clark's Mathematical Tables, p. 33.

1.431. INTERPOLATION BY SLIDE RULE: The calculation is greatly facilitated if the logarithms are taken from a slide rule: that described in **4.11** can be adapted to this purpose by having a uniform decimal scale corresponding exactly with the unit logarithmic scale, thus making stationary scales (**5.5** f. n.) on which opposite graduations give a number and its logarithm. This would correspond on the ordinary slide rule (not described in this book) to having a pointer attached to the cursor exactly opposite the cross line so as to show, on a 10 inch scale set to correspond, the logarithm of the number under the cross line. In some slide rules the 25 centimetre scale can be easily adjusted to this purpose; but another device for giving the logarithms on the back of the slide seems to have been found more satisfactory.

Note—A prominent physicist says that the slide rule should not be used for simple laboratory experiments: there is no instrument, he declares, which the student is more liable to abuse. (His reason for saying this is probably that in the experimental sciences good measurement should make the computer unnecessary.) Test the accuracy of logarithms obtained from the slide rule.

1.5. REGIONS: A simple curve divides the plane in which it lies into two regions, points in which have opposite characteristics in relation to the algebraic equation which the curve represents. Thus if the line, $3x-y+6=0$, is drawn and on the same figure points are marked, it will be found that the coordinates of all points on one side of the line, when substituted in the expression on the lefthand side of the equation, give a positive value; those of points on the other side, a negative value. And so the two sides of the line may be described as *positive and negative respectively*. The length of the *perpendicular* from a point (x, y) to the line varies as the value of $3x-y+6$, and thus perpendicular lengths may be regarded as having signs.

A similar statement is obviously true for the circle $x^2+y^2-a^2=0$ also; the expression on the left with coordinates of any point substituted in it is called the **power** of the point with respect to the circle; this "power" is positive or negative according as the point is outside or inside the curve. When positive, it is the square of the *tangent* from the point to the circle.

Similar statements, though they have not such simple geometrical interpretations, are true for the parabola, the ellipse and other curves[1] and can easily be verified by trial.

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1 *v.* Smith's "Conic Sections", Arts. 82, 92, 113.

Even for the hyperbola $xy=1$ which appears to divide the plane into three regions the statement holds good; the first and the third quadrants are each divided into regions in which $xy<1$ and $xy>1$, and in the second and fourth quadrants $xy<0$: but all this may be combined in the one statement that "between" the branches of the curve $xy-1$ is negative, and "beyond" it is positive.

The idea of regions which are bounded by curves, in crossing which a definite change takes place, is very common. In physics we have curves defining regions which represent conditions under which a substance may or may not be gaseous[1]. In medical science there are dissociation curves, which indicate under what conditions and to what extent oxygen may be combined with hæmoglobin. In Bainbridge and Menzies' "Essentials of Physiology", p. 257 this idea is emphasised by colouring the region on one side of the curve red, on the other side blue. In Joslin's "Diabetes Mellitus"[3] p. 389 arrows are drawn away from the dissociation curve of normal CO_2 (cf. **9.2**) in an excessively simple diagram, and are labelled "alkalosis" and "acidosis" to indicate the respective dangers in too great departures from this line.

1.6. LINKAGES for connecting circular and rectilinear motions are an illustration of a particular case of **inverse curves.** Each of two curves is said to be the inverse of the other with respect to a *centre of inversion* O when *every line through this centre cuts the curves at distances from* O *whose product is constant.* Such curves may be plotted point by point, (fig. 5), or they may be traced by two points on a system of rods jointed together in special ways: these are called *linkages.* They may be adapted to connect parts of a machine which are to have inverse motions, *e.g.*, motions in a circle and in a straight line. For it is easily proved that the inverse of a straight line is a circle through the centre of inversion.[2]

1 *v.* Watson's "Textbook of Physics"[17] p. 275, Senter's "Physical Chemistry," p. 180, etc.

2 See any book on geometry, especially Workman and Cracknell's Geometry p. 474. Portions of linkages used for demonstration purposes may be identified at the foot of the heavily-laden figure 2.

PEAUCELLIER'S CELL (fig. 6) consists essentially of six links, four forming a rhombus, and two other equal rods, either longer or shorter than the four, jointed as shown in figure 6. But it is convenient to attach an extra rod to one of the tracing points, so that by fixing a point in the rod a circle through O, the fixed centre of inversion, is described by this tracing point. For this linkage it is easy to prove that

$$OP.OQ = OA^2 \sim AP^2.$$

HART'S LINKAGE (fig. 7) is simpler in construction, being simply a **crossed parallelogram.** It permits of more adjustment than Peaucellier's Cell, for the centre of inversion may be at any point on one of the arms, say AD, and then the tracing points P and Q are where the line through O parallel to the *parallel diameters* cuts AB, CD, the arms which differ in length from the arm on which O is. The proof of the inverse property is somewhat more difficult than in the preceding case and is given here. In figure 7 we have, if $AO : OD :: m : n$, and $AH \perp BD$,

$$OP.\ OQ = \frac{n}{m+n}AC.\frac{m}{m+n}BD = \frac{mn}{(m+n)^2}(DH+BH)(DH-BH)$$
$$= \frac{mn}{(m+n)^2}(AD^2 - AB^2)$$

Ex. Repeat this proof with O on AB. What results from changing the position of O on an arm?

Note—A description of a simple linkage with geometrical interest will be found in the Philosophical Magazine, 1918, **36** 143.

1.7. BINOMIAL COEFFICIENTS. Knowledge of the binomial coefficients of the expansion of $(a+b)^n$ is often required. They can easily be calculated from Newton's formula ${}_nC_r/r!$; but for some it may be useful to have them set out as in **Stifel's Table**[1] one of the earliest forms in which they were presented.

I														I					
I	I												I		I				
I	2	I										I		2					
I	3	3	I								I		3		3				
I	4	6	4	I						I		4		6					
									I		5		10		10				

1 This table was known to the Chinese about 1300. (Nunn's "Teaching of Algebra", p. 212.) It was Stifel who announced the end of the world for October 3, 1533! *v.* Ball's "History of Mathematics", p. 216.

The rule for compiling the table is obvious—fill each place with the sum of the number above and the number to the left of that number. If the table is arranged symmetrically as on the right, the rule simplifies to—each number is the sum of the two numbers immediately above it on either side; and the obvious symmetry makes it unnecessary to write more than half the table. For reference purposes this table should be completed up to $n=20$.

Ex. Write out with the help of Stifel's table the expansions of expressions like the following, $(a+2b)^9$, $(\frac{1}{2}a+5b)^{11}$.

1.8. DETERMINANTS. It has been said, in apparently irreverent paradox, that Mathematics is the science in which we do not know what we are talking about, and in which we do not care whether what we say is true or not! Put positively, this means that Mathematics is the science of symbol and its rational use: the symbols are defined with precision, but it is no concern of the mathematician as such to know whether the definition corresponds to anything in the external world, or not.[1] His work is to unfold the consequences of the definitions he has formulated, whether or no these results have intelligible application to the world of things.

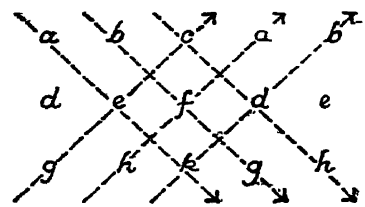
One of the most striking examples of this is the *generalised geometry* (*i. e.*, geometry of many dimensions—more than three !) which found an interpretation through the theory of relativity after more than half a century. Equally striking and more intelligible, is the use made of *"imaginary" quantities* in finding the explanation of phenomena in, say, electricity. It would take us rather far to attempt to explain "imaginaries", though the idea is not difficult. We shall take here only some of the simpler facts about another symbol which has immediate applications: it is called a **determinant;** for one of its properties is that it determines the relation between the coefficients of expressions which

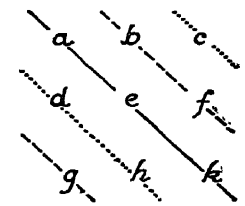
1 Cf. the astonishing result, which we may write as $-\infty=+\infty$, got when we play with the changes in the value of y as x increases through zero on the hyperbola $xy=1$.

have certain characteristics *e.g.*, linear equations which are consistent[1] (cf. **1.83, 1.84**).

1.81. A determinant is an array of n^2 quantities, called **constituents**[2] arranged in a square: this symbol represents the sum of all the products ($n!$ in number) of these quantities, taking for every product one and only one constituent from each row, and from each column: the sign of the product is settled by a rule which for determinants containing no more than three constituents in a row or column (*i.e.*, **third order** determinants) is very easily expressed thus: Re-write the first two columns after the third; then the six possible products are on the complete diagonals through these five columns; the positive sign is attached to the products downwards from the left, the negative to the others.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$





The determinant here shown represents

$aek+bfg+cdh-gec-hfa-kdb$. After a little practice it becomes unnecessary to rewrite the columns; the broken diagonals may be picked out as indicated on the right for the positive products. The diagonal downwards to the right through the leading constituents is called the **principal diagonal**, the other the *secondary diagonal*.

Ex. 1. Find the values of

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}, \quad \begin{vmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{vmatrix}, \quad \begin{vmatrix} 1 & \cos A \\ \cos A & 1 \end{vmatrix},$$

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}, \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

---

1 Thus $\left.\begin{matrix} a\ x - b\ y + c\ z = 0 \\ a_1 x + b_1 y + c_1 z = 0 \end{matrix}\right\}$ are consistent if $\begin{vmatrix} b & c \\ b_1 & c_1 \end{vmatrix} : \begin{vmatrix} c & a \\ c_1 & a_1 \end{vmatrix} : \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = x : y : z$

note the cyclic order.

2 These are sometimes called **elements**, but some writers apply this word to the products which are summed.

Ex. 2. Show by expansion that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{vmatrix}$$

1.82. Elementary properties of determinants are

(i) An **interchange of rows and columns** has no effect on the value.

(ii) If **any two rows** are **interchanged,** the sign of the determinant is reversed. Consequently, the value of a determinant is zero if any two rows or columns are identical.

(iii) Multiplying the determinant by **a factor** means multiplying each constituent of one row by that factor. Accordingly, if two rows are made identical by putting $a=b$, then $a-b$ is a factor.

The truth of these and other properties can easily be verified.

Ex. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

1.83. The use of determinants in **solving simultaneous equations** has been referred to already on p. 2, I; for the solution of

$$\left.\begin{aligned} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \end{aligned}\right\}$$ can be written $$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{\mathrm{I}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Similarly for three unknown quantities, the solution[1] of

$$\begin{aligned} a_1x+b_1y+c_1z+d_1=0 \\ a_2x+b_2y+c_2z+d_2=0 \\ a_3x+b_3y+c_3z+d_3=0 \end{aligned}$$ is $$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{y}{-\begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix}} = \frac{-\mathrm{I}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Ex. Test this by solving the equations

$$x+2y-z=6,\ 2x-y-3z=3,\ 4x-2y-2z=2.$$

1 The negative sign of 1 has to be introduced for determinants of odd order.

1.84. The homogeneous quadratic expression, $ax^2+2hxy+by^2$, we know, is a perfect square if $\begin{vmatrix} a & h \\ h & b \end{vmatrix}=0$; and it cannot be factorised if the value of the determinant is positive. Compare this condition with the expression itself written as

$$(\mathbf{a}x+\mathbf{h}y)\ x + (\mathbf{h}x+\mathbf{b}y)\ y.$$

It can also be proved that the general quadratic expression, $ax^2+2hxy+by^2+2gx+2fy+c=0$,

which can be written $\left\{\begin{array}{l} (\mathbf{a}x+\mathbf{h}y+\mathbf{g})\ x \\ +(\mathbf{h}x+\mathbf{b}y+\mathbf{f})\ y \\ +\ \mathbf{g}x+\mathbf{f}y+\mathbf{c} \end{array}\right.$,

can be expressed as a product of **linear factors** if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$

Ex. 1. Test x^2-y^2+2y-1, $2x^2+7xy-4y^2+x+13y-3$ for linear factors. Find these. Construct other such examples.

Ex. 2. For what values of h have $12x^2+2hxy+2y^2+11x-5y+2$ and $2hxy+5x+3y+2$ linear factors?

Ex. 3. Modify this idea about the condition for factors so as to make it apply to the homogeneous quadratic function $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$.

1.85. One of the most useful applications of determinants is to find the **area of triangles** whose coordinates, for equal scales along the axes, are given. The **area of the triangle** whose coordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3) is $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

This can be verified easily for
(2, 0), (2, 5), (3, 5);
(3, 0), (1, 7), (−5, 0); etc.

It is an interesting consequence of the way in which we usually arrange the coordinate axes that, if the vertices are taken in the clockwise order in the rows of the determinant, the sign of the area is negative. Verify this.

Ex. Verify by taking simple instances that the area of the quadrilateral formed by (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) is $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_2-x_4 & y_2-y_4 & 0 \end{vmatrix}$

1.9. CIRCULAR MEASURE: A right angle, being the quarter of a complete revolution, is a natural unit in which to measure other angles. But the subdivision of the right

angle is arbitrary, and in any case there is no simple way of connecting these measures of an angle with the lengths of the lines that contain the angle. It is easily proved that angles at the centre of a circle are proportional to the arcs they are subtended by at the circumference; and that for one angle the ratio, *arc*: *radius* is constant, whatever the size of the radius be. If this ratio be taken as the measure of the angle, then the unit angle, the **radian**, is the angle subtended by an arc equal to the radius, and this is the unit in "circular" measure. This system has great advantages over the right angle system for many purposes, one of which is pointed out in **3.13, 3.23**; cf. also **9.61.** But these advantages, and the use of the formula $\mathbf{arc} = \mathbf{r}\theta$, are fully explained in text books on Trigonometry.

In reference to **7.11** mention may be made of circular diagrams[1] which are extensively used to show the proportions of different constituents in a whole. If the whole circumference of a circle, *i.e.*, $2\pi r$, be taken as 100, the percentage of each constituent can be marked off on a proportionate arc, and the points of division joined to the centre. If by colour, or otherwise, there be an easy way of making clear what each sector refers to, this makes a compact and striking diagram. But its use is seldom more than pictorial. A good example of this type of diagram, where owing to the large scale it has been possible to convey much detailed information abont castes in the Bombay Presidency, may be found in the Census of India, 1921, VIII 188. In Whipple's "Vital Statistics", p. 172 a series of such diagrams is given to convey similar information about the U. S. A.; but in such a case ordinary columns or bars would have been more effective in showing changes. The ratio of the totals represented by different circular diagrams is that of the squares of the radii: this fact makes for compactness of representation.

1 These are called "Pie" diagrams; but surely not for the specially American reason hinted at in Pearl's "Medical Biometry," p. 110! A suggestive name for these diagrams is *component charts.*

CHAPTER II

TYPICAL GRAPHS

2. PLOTTING CURVES: You have learned already in studying algebra how relations like $4x+7y=3$ and $y^2=6x$. which are true for any number of pairs of values of x and y, may have their general character made clear in a graph; the pairs of values of x and y which satisfy them are represented by points which lie in a definite line. This mode of representation may be extended to **any relation whatever between two variables,** *e.g.*, $9y=\sin x$, $6y^3=4x^2-x$, $y=5^x$; for all that is required to make this possible is that we find out in any way pairs of corresponding values of x and y.

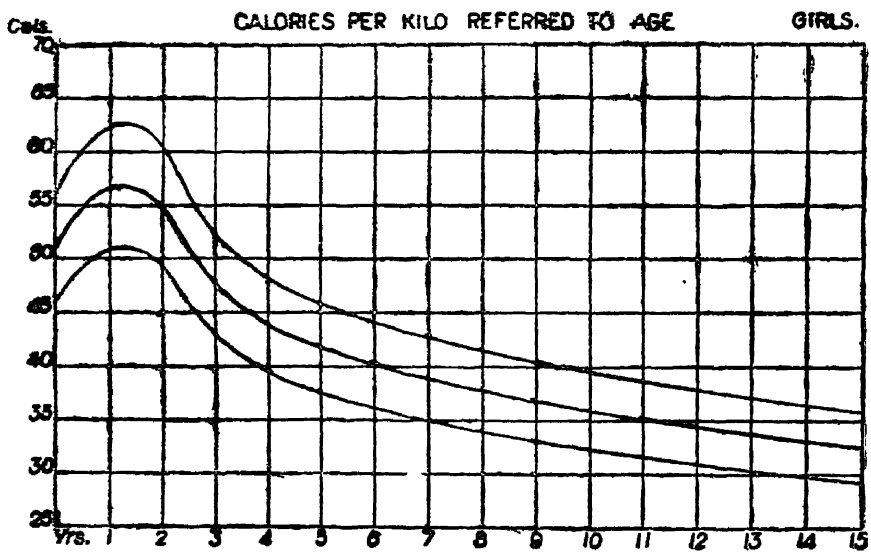


Fig. 8. "Basal metabolism, calories per kilogram of body weight for twenty-four hours of girls at different ages. The curve is projected from twelve years upward. (Talbot)."

Ex. 1. Construct a curve from any convenient table of values, *e.g.*, from Knott's Mathematical Tables p. 30, col. 8 draw $y=\sqrt[3]{10x}$, taking $x=0$, 10, 20,.....100 Insert additional values where you have doubt as to how the curve runs. It is useful to study similarly the graph of $y=\log x$; cf. **1.32** Ex.

Ex. 2. Sketch the curve $y = \frac{2x}{x^2+1}$.

This is given in Lamb's "Infinitesimal Calculus",[3] p. 27: cf. typical curves showing the production of heat in the human body, fig. 8, from Du Bois' "Basal Metabolism", p. 121 (Lea & Febiger, New York). Here we have a family of curves (cf. **2.13** below), whose equation may be $y = \frac{kx}{x^2+1}$, where k is given different values. By trial approximate to the value of k for one of these curves. (The origin for this equation must be taken to the left of that shown in the figure.)

Cf. also the graph of $y = \frac{x^2-x+1}{x^2+x+1}$, given in Davison's "Elementary Algebra", fig. 20. Try the effect of changing the origin (**1.21**) of this curve to (0, 1).

Ex. 3. A line AB of fixed length is divided by a variable point P. Denoting AP by x, and the ratio $AP:PB$ by y, plot the curve which shows the relation between x and y. Paying respect to the signs of AP/PB, do this further when P divides AB externally, and find the equation to the complete curve. What kind of curve is it?

Ex. 4. On the lefthand line in figure 1 are three scales, two for ages of males and females which are fitted to a regular scale for "normal basal metabolic rates" N at the corresponding ages. Draw two curves showing the variations of N with increasing age for men and for women: cf. figure 8, and **5.31**.

PARABOLIC CURVES

2.1. INTEGRAL INDICES: One of the simplest types of relations between variables is represented by $y=x^n$, where n may have any value, integral or fractional, positive or negative. Plot roughly a few of the simpler curves of this type, *e. g.*, $y=x$, $y=x^2$, $y=x^3$, $y=x^4$, etc. It is readily seen that they fall into two classes according as n is **odd** or **even**—a very important distinction. Thus, for k positive, the graph (or curve) for $y=kx^{2s+1}$ extends indefinitely towards the upper righthand and the lower lefthand corners of the figure, *i. e.*, it lies in the first and third quadrants.

Ex. 1. Modify this last statement to make it apply to the case of k negative.

Ex. 2. Make similar statements for the cases when $n=2s$. What is the curve corresponding to $n = 0$?

Ex. 3. Trace the changes in the shape of successive odd or even curves as the index increases. What is the limiting shape of the curve in each case when n becomes very large?

2.11. INTERCHANGE OF AXES: If n is the **reciprocal** of an integer r, *i.e.*, if y is the r^{th} root of x, the calculation of values of y from those of x may be difficult. But $y=x^{1/r}$ may be written $y^r=x$, and then x is easily calculated in terms of y. When the points are plotted, curves with the same

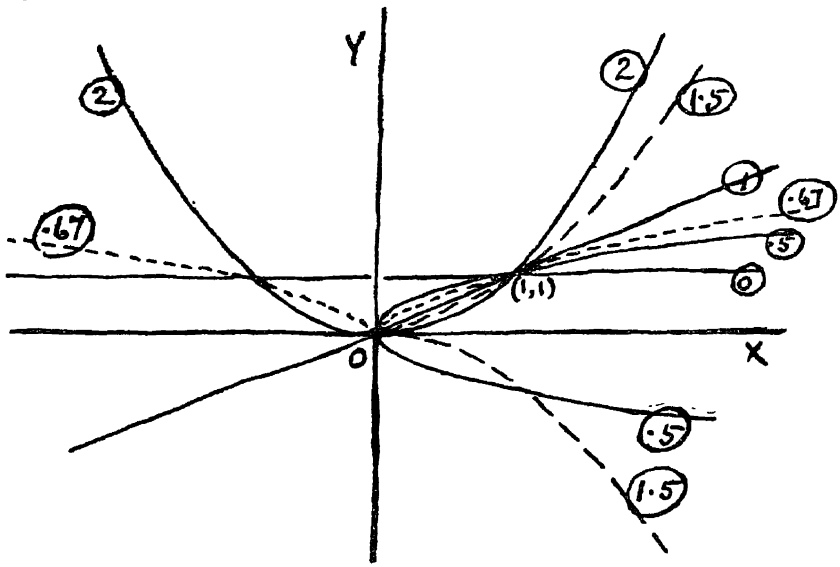


Fig. 9, $y=x^n$.

characteristics as before are obtained, the only difference being that the axes are interchanged. The correspondence of this with the algebraic change is obvious: if (p, q) is on $y=x^n$, then (q, p) is on $y^n=x$.

CONSEQUENT SYMMETRY: Another important geometrical property can be derived from this. If $y=x^n$ and $y^n=x$ are, as we have just seen, respectively related in the same way geometrically to the x axis and to the y axis (or conversely), *i.e.*, if there are corresponding points throughout these curves which are the extremities of equal perpendiculars drawn in, say, the positive direction for the respective axes; then they are both related in the same way to the bisector of the angle between the axes. We put this more concisely in saying that these curves are symmetrical with respect to the line $y=x$, when the units of measurement along the axes are the same.

Ex. Put the above general statement about identical relationship of curves to the respective axes in a more particular form for the curves for $y=x^{2/3}$ and $y=x^{3/2}$ shown in figure 9. How must this figure be modified so as to show these curves as symmetrical with respect to $y=x$?

Note : $y=x$ gives the members of the families of curves $y=x^n$ and $y^n=x$ which are coincident.

2.12. AXIAL AND CENTRAL SYMMETRY: The curves $y=x^n$ illustrate symmetries of two kinds. There is symmetry with respect to a line, the **axis of symmetry,** as seen above in $y=x^{2s}$, where s is a positive integer ; also with respect to a point, called the **centre of symmetry.** This occurs in $y=x^{2s+1}$; the centre is here the origin. The geometrical tests for these forms of symmetry are alike. For the first, a perpendicular drawn from any point on the curve and produced its own length beyond the axis reaches the curve in the symmetrical point. For the second, a line joining any point on the curve to the centre of symmetry and produced its own length beyond the centre reaches the curve again in the point centrally symmetrical to the first.

Ex. Test these statements in particular cases, *e. g.*, $y^3=3x$, $y=2x^4$. Are they still true when the scale along the y axis is double that on the x axis ?

2.13. THE FAMILY OF CURVES: To study the curves where n is *any* fraction it is best to think first of the region between the origin and the point (1, 1). Regarding the curves we have already considered in **2.1, 2.11** where n was successively equal to the increasing numbers $0,\ldots\frac{1}{2}$, $1, 2, 3,\ldots$, a continual change is readily seen: these curves come successively closer to the x-axis. We have only to take the trouble of working out the position of a few particular points to assure ourselves that curves for which the index of x is intermediate between the values given above have an appropriate geometrical property; they lie between the corresponding curves in the figure.

Ex. Taking 5 ins. as unit, draw carefully the curves $y=\sqrt{x}$, $y=x$, $y=x^2$ between (0,·0) and (1, 1). Show that the points $\left(\frac{1}{8}, \frac{1}{4}\right)$, $\left(\frac{8}{27}, \frac{4}{9}\right)$, $\left(\frac{27}{64}, \frac{9}{16}\right)$ are on the curve $y=x^{2/3}$, and $\left(\frac{1}{16}, \frac{1}{64}\right)$, $\left(\frac{4}{9}, \frac{8}{27}\right)$, $\left(\frac{25}{36}, \frac{125}{216}\right)$ on $y=x^{3/2}$. Plot such points as these and show that they lie on curves intermediate between the three first drawn. Verify by use of logarithms that the points given by these formulæ for values such as $x=·345$ lie accurately on these curves.

The extension of these curves into the region beyond (1, 1) is easy. As may have been expected there is no

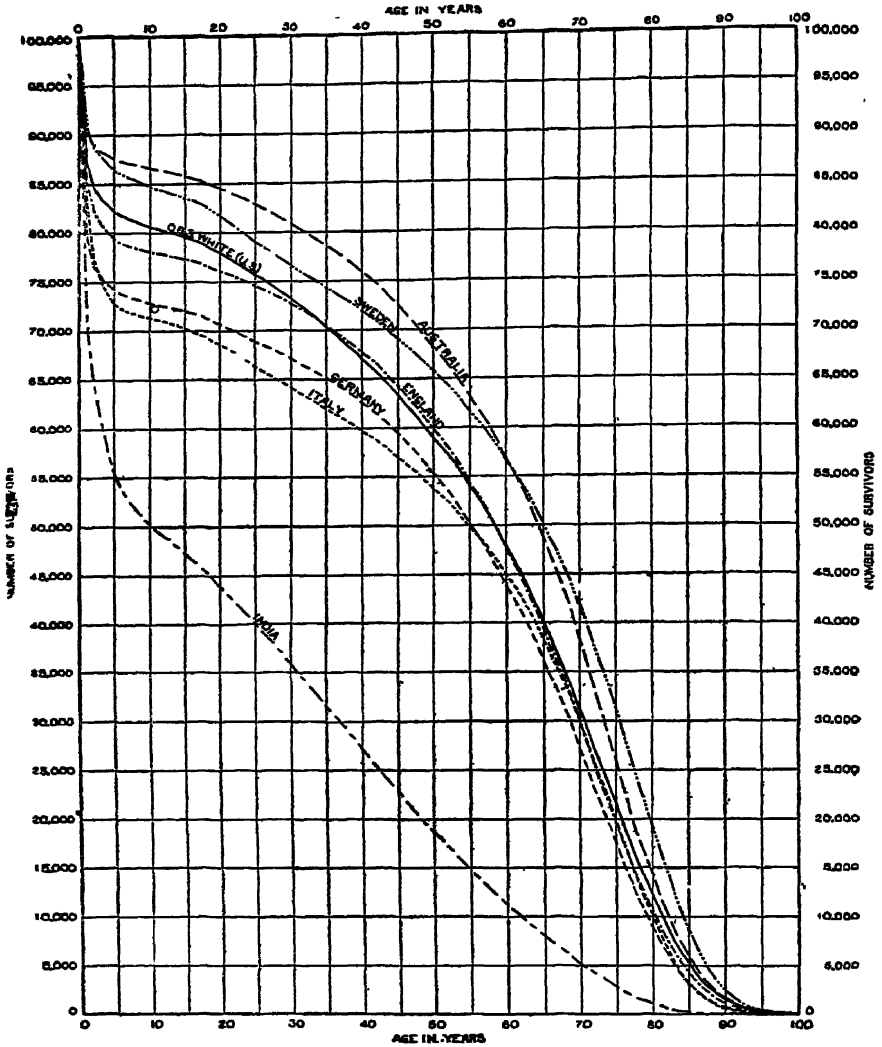


Fig. 10. "Survivor" Curves (O. R. S. signifies Original Registration States.)

change in the relations between the curves; only their order with respect to the x axis is reversed.

Tests of this statement will readily suggest themselves to the student, *e. g.*, whether in the figure drawn in the above example the values of y corresponding

to, say, $x=1{\cdot}234$ lie on the extensions of the respective curves. A test should also be applied to a new figure with a smaller scale.

Ex. By means of a table of logarithms make a table of values of $y=2x^{2/5}$ at intervals of 0·333 between $x=0$ and $x=4$. Plot these values, and show that the points obtained lie on a continuous curve.

These curves can also be extended into the regions where x and y are either one or both negative. There is no great difficulty about this provided it is remembered that an even root of a negative number cannot be represented in this figure. Interesting modifications of the curves occur, but it does not serve our present purpose to trace these out, though for problems like those dealt with in **9.4** it is important to be able to recognise the types of these curves.

Further information may be fonnd in Barnard and Child's "A New Algebra", pp. 493-5; also in Gibson's "Elementary Treatise on the Calculus," p. 45, or in his more elementary works.

2.131. It is frequently more important to think of such curves as expressing a relation within a **restricted range** of values of x. Thus in the very striking figure, 10, we have curves which for most of their length suggest members of the family $y=x^n$, save that they are reversed with respect to the y-axis, and the origin is not what it was in the simplest figure (**2.1**). The contrast expressed numerically is that for India the value of the index is greater than unity, for countries more fully under the influence of western civilisation the index is less than one.

If it be desired to test a formula on these curves, a suitable formula may be obtained with the help of the rule for change of origin (**1.21**) as $y+(x-1)^n=0$ This for the "India" curve corresponds to a value of n somewhat greater than 2, *e. g.*, the point $(\frac{1}{2}, \frac{1}{4})$ on $y=x^2$ corresponds to (50,.25,000) which is considerably above the curve in the figure, the unit for x being 100 years, that for y 100,000 persons: cf. also **9.41**. In **3.213** these survivor-curves are converted into mortality-curves.

The diagram is taken from Pearl's "Medical Biometry and Statistics", p. 184 (originally from J. W. Glover's U.S.A. **Life Tables**); this should be compared with the apparently simpler curves on p. 191 of the same work; see also **9.3**. Another example of such curves occurring in experimental work is the dissociation curves of oxyhæmoglobin, which show to what extent oxygen and hæmoglobin remain united under different circumstances. (*v.* Bainbridge and Menzies', "Essentials of Physiology", pp. 258, 259, Longmans). Cf. p. 15.

2.14. DISCONTINUOUS CURVES. When *negative* values of n are considered, it is at once seen on sketching such

graphs as $xy=1$, $x^2y=1$, $x^3y=1$ that there is a great change in the character of the curves : none of them passes through the origin, and they are all in two parts, one for x positive, and the other for x negative. But that these still belong to **the same family** of curves is evident from the persistence of the division into two classes according as n is odd or even :

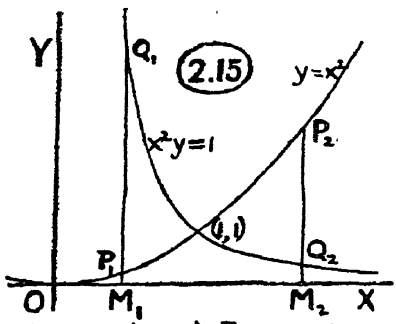


Fig. 11. Axial Inversion.

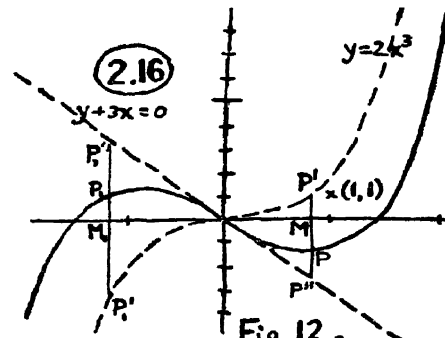


Fig. 12.

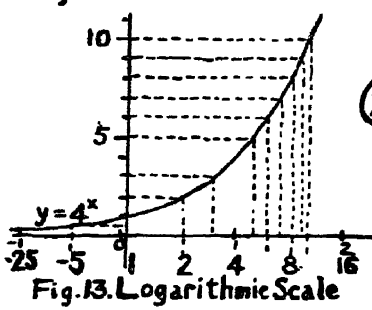


Fig. 13. Logarithmic Scale

2.22

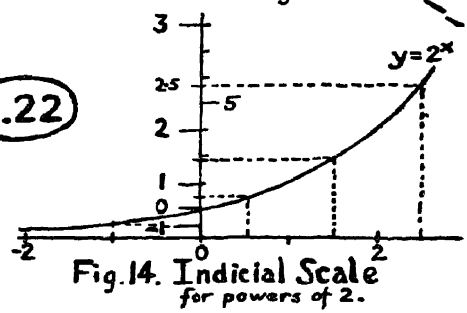


Fig. 14. Indicial Scale for powers of 2.

for n odd, the curves lie in the first and third quadrants, pass through (1, 1) and (−1, −1), and are symmetrical with respect to the origin; for n even they lie in the first and second quadrants, pass through (1, 1) and (−1, 1), and are symmetrical with respect to the y axis.

Ex. 1. Sketch the graphs of the hyperbolas $xy=3$ and $xy=\frac{1}{3}$. Do these curves possess all the symmetries of $xy=1$? If so, why ? [Note that symmetry with respect to bisectors of the angles between the axes depends on the position of (1, 1).]

Ex. 2. Discuss the " behaviour " of $y=x^0$ in respect of the points it passes through. Do (0, 0) and (0, 1), or indeed any point of the y-axis, lie on the " curve " ? (Examine carefully figure 9.)

Here again we can illustrate the form of these curves from curves obtained in investigations beyond the scope of geometry. In physics the relation between the pressure and volume of a gas (Boyle's law) is so expressed; in economics constant outlay curves, which represent an increase in demand proportional to the fall in price, are of this form. (*v.* Marshall's " Principles of Economics ", p. 840.)

2.15. INVERSE RELATION. A **geometrical interpretation** (cf. **2.11**) can easily be given to the connection between the equations $y=x^n$ and $x^n y=1$. If the ordinate through any point M (fig. 11) on the x axis cut the corresponding curves in P and Q, we have $MP.\ MQ=(OM)^n.\ (OM)^{-n}=1$. This holds true for all values of x and n, due regard being paid to signs. Thus a relation analogous to that between inverse curves (**1.6**) holds between simple parabolic[1] curves in which indices of x are equal but of opposite sign—the product of the distances from the x axis of points in which the curves are cut by any ordinate is constant.

Note: This constant need not be unity; for $y=k_1x^n$ and $yx^n=k_2$ the product is k_1k_2.

2.16. COMPOUND CURVES: The reference to *simple* parabolic curves implies that they may be combined. This is effected by simply adding for two or more curves the ordinates corresponding to the same abscissæ. The results are full of interest. Thus, corresponding to the addition of terms in algebraic expressions there is a geometrical process by which a curve can be found to represent any equation of the type

$$y = ax^n + bx^{n-1} + \ldots\ldots + vx + w.$$

Take as a simple example $y=2x^3-3x$. The curves corresponding to the terms of the righthand side separately are shown in figure 12 by broken lines. The continuous curve is got by taking for every abscissa the sum of the two corresponding ordinates to give the ordinate of the compound curve, *i.e.*, $MP' + MP'' = MP$.

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1 " Parabolic " when strictly used, does not apply to discontinuous curves; it refers only to curves for which the index of x is positive and rational. The title **Hyperbolic Curves** might have been inserted before **2.14**; but it seems unfortunate to make a distinction when the fundamental contrast between odd and even indices remains un-

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If $y = 3x$ had been drawn instead of $y + 3x = 0$, the same result would have been got by measuring back from P' a length equal to the corresponding ordinate to $y = 3x$, due regard being paid to sign.

The curve might have been got directly also by calculating a series of values of $2x^3 - 3x$.

This example has a special interest in that, since all the terms are of odd degree, the equation remains true if $-x, -y$ are substituted respectively for x, y throughout. The corresponding geometrical property is that the curve is symmetrical with respect to the origin (cf. **2.12**); this is evident from the figure.

Whether there be symmetry or not, this process can be carried out and curves of a great variety of shapes built up.

Ex. 1. Sketch the interesting parts of the graphs of

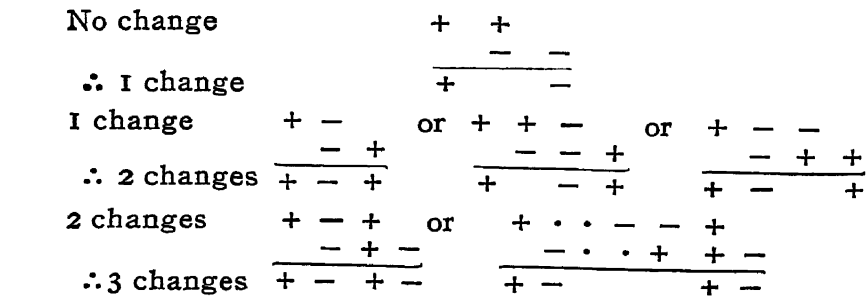
$$y = x^3 - 6x^2 + 11x - 6,$$
$$y = x^3 - 7x^2 + 16x - 12.$$

What geometrical meaning has the fact that the righthand side in each of these can be factorised? What are the equations for these graphs when the origin is at (2,0)?

Ex. 2. Draw the graph of $y = x^2 + 2/x$, and find the roots of $3x = x^3 + 2$; also a root of $2x = x^3 + 2$, correct to the second decimal place. Are there more than one root of this second equation?

Ex. 3. With respect to the same axes draw the graphs of $y = x^5$ and $y = 5x - 10$. From the figure find the roots of $x^5 - 5x + 10 = 0$ and $x^5 - 5x + 1 = 0$. Verify your results by substitution. (Cf. **1.211**).

2.17. LINEAR FACTORS AND CHANGES OF SIGN: A factor $x - a$ in a polynomial signifies that a is a root of the equation got by equating that polynomial to zero. There is a change of sign in the expression $+x - a$, and as often as any polynomial is multiplied by this binomial, at least one more change of sign is introduced. This can be tested in simple cases and will be seen to be true generally:



The signs of the sums of unlike pairs of terms do not affect the result and are not inserted in the third lines. Missing terms are represented by dots. In the case of

$$\begin{array}{ccccccccc} + & + & + & + & + \\ & - & - & - & - & - \\ \hline + & & & & & & & & - \end{array}$$

as many as five changes *may* be introduced by the multiplication, but we can be certain of only one of these.

The converse of this gives the very useful **Rule of Signs** (due to Descartes) that the number of positive roots of an algebraic equation cannot exceed the number of changes of sign in successive terms.

This rule can be adapted to negative roots by changing the sign of x throughout the equation, and then considering the number of changes of sign in successive terms to give the maximum number of negative roots.

Ex. 1. Find from a graph or otherwise the roots of
$x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$.

Ex. 2. Find the roots of $4x^5 - 23x^4 - 3x^3 + 28x + 12 = 0$,
and of $4x^5 - 23x^4 - 3x^3 + 28x + 8 = 0$.

218. METHOD OF DIFFERENCES: Mention is made here of an important method which is based on the fact that differences such as $(x+1)^2 - x^2$, $(6x+1+8)^5 - (6x+8)^5$ are of degree less by one than the expressions whose difference is found. Thus, if we obtain the *differences between successive values* of any rational algebraic function taken at unit intervals of the independent variable, we see that they are all *of degree at least one lower* than the original function. Repeat this process on the differences obtained, and we get the *differences of the second order*, or simply *second differences*, denoted by Δ^2; and so on. It may happen that we get the n^{th} differences Δ^n all alike (and therefore the $(n+1)^{\text{th}}$ differences Δ^{n+1} zero), i.e., the n^{th} differences represent a constant function of degree zero. Counting back we see that the original function, Δ^0 as we may represent it, must have been of degree n.

Note that Δ here represents an *operation* just as D does in **3.1**, and the index shows repetition of this operation, which is the fundamental one in the *Calculus of Finite Differences*. The similarity of the result to that in **3.13** (i) is evident.

Ex. 1. Show that the third differences in a table of cubes are constant.[1] Test the method also on successive values of functions you have already calculated in, e.g., **2.16** Ex. 1.

Ex. 2. The fact that successive differences of the tabulated values of logarithms, etc. have their first differences constant has been employed in **1.42** to treat these values as if the points representing them lay on straight lines. Show that the points representing the logarithms of 10, 11, 12, 13, 14,..... ...may be regarded as lying on a curve of the fourth degree. Test other parts of the table of logarithms, and other tables, in a similar way.

(This suggests, what is a fact, that functions like log sin$x$ may be represented with a certain degree of accuracy by a series of algebraic terms involving powers of x up to a certain degree—a **power series**, it is called.)

Ex. 3. Test series of values observed at regular intervals (such as those of B in **1.33** Ex. 2) to see if any light is thrown by this method on the nature of their variation.

EXPONENTIAL CURVES

2.2. FORM OF THE CURVES: $y=n^x$ gives graphs that are very different from those given by $y=x^n$. Their general character is easily seen by putting n equal to, say, 4, and plotting a series of points. No part of the curve is below the x axis, and it extends from the distant left, **always rising**, slowly at first to (0, 1), and thereafter more and more rapidly as it continues to the right without limit (fig. 13).

By giving a succession of positive values to n it can be seen that these characteristics persist for all the curves. If n lies between 0 and 1, *e.g.*, $\frac{1}{4}$, we get mirror-images in the y axis of the above curves.

For negative values of n the curves are not continuous; but this case is not like that of **2.14**.

Ex. What are the graphs for the boundary values of n, viz., 1 and 0? Compare these with the boundary curves of the classes of $y=x^n$.

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1 Note that this gives an easy method of constructing a table of cubes.

2.21. PROPORTIONAL GROWTH (OR DECAY): These exponential curves are sometimes called growth curves, or, more particularly, compound interest curves. (Cf. Griffin's "Mathematical Analysis," pp. 236 ff.; also Nunn's "Algebra," I p. 269, etc.) The reason for this cannot be given precisely, until we come to the next chapter (**3.13** ii). But, if we consider values of y at equal intervals h of the x variable, we find that their ratio is the same at all parts of the x scale *i.e.*, $n^{x+h}/n^x = n^h$, which is independent of x. And the correspondence of this n with $1+r$ in the formula $A=P(1+r)^n$ for the amount A in n years of a sum of money P at a rate of interest r per annum is easily seen (**1.43**). The general idea is that the amount of growth is proportional to the absolute size of the thing growing, and in this curve it is clear that the rate of increase (*i.e.*, the slope) is greater the greater the ordinate becomes.

Ex. 1. On a convenient scale, say 1 cm. = 1, the same for both axes, draw the graph of $y=2^x$ between $x=-3$ and $x=3$. By the method given in **1.32** draw tangents to the curve at the points whose abscissae are $-2, -1, 0, 1, 2$. Find the tangents of the angles θ made by these tangent lines with OX, and evaluate in each case $\frac{\tan\theta}{y}$; What do you notice about the values you get? (For use later this graph should be constructed carefully on squared paper: cf. **3.130**).

Ex. 2. Show that only one curve of equation $y=ka^x$ passes through two given points (x_1, y_1). (x_2, y_2): draw this curve in some particular case. Show how this fact is connected with the possibility of finding in one way only a certain number of geometric means between two quantities y_1 and y_2; cf. **1.43**. Relate this to the similar possibility for arithmetic means.

2.22. FUNCTIONAL SCALES: The equation $y=4^x$ can be written $\log_4 y = x$, and an accurate graph of the equation could be used to find the product of numbers represented by ordinates y by adding their corresponding abscissae and taking the ordinate corresponding to this sum: thus

$$x_1+x_2 = \log y_1 + \log y_2 = \log y_1 y_2.$$

Ex. Express this in the exponential notation with which we started here.

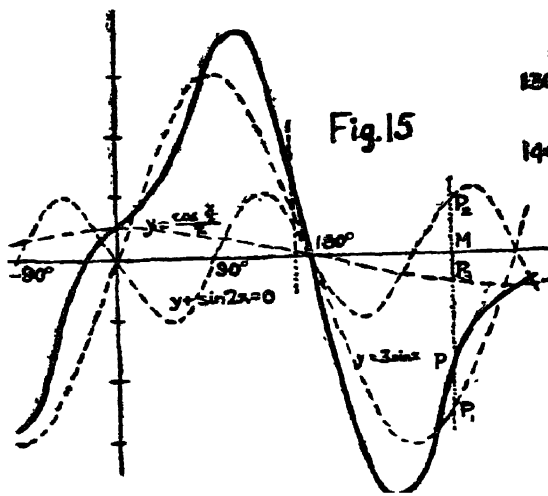
But more important is it to note that we may write on the x axis the corresponding values of y and so get a **logarithmic scale**, *i.e.*, a scale such that each number on it is at a distance from the end of the scale proportional to its

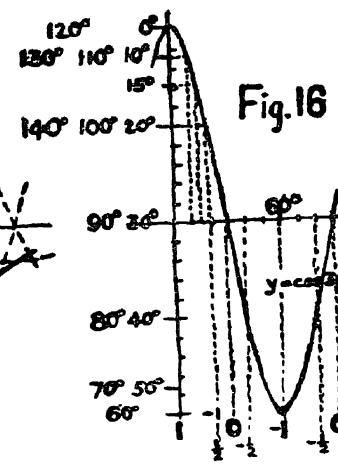
logarithm. This is shown in figure 13, where the logarithmic base is 4. This scale is just such as is used in the slide rule, where however the base taken is that of common logarithms, 10.

It should be noted that this device may be applied to all graphs to give functional scales of any kind. Thus on the y axis of the curve $y=2^x$ shown in figure 14, there is marked, more clearly than the uniform scale, the **indicial scale** of powers of 2 which equal the distance of the graduations from 0.

Ex. 1. Obtain from the graph of $y=x^2$ a scale of squares and one of square roots.

Ex. 2. From $y=x^{-1}$ get a scale of reciprocals (4,3).





The characteristics of the logarithmic scale should be carefully noted: the graduations of the scale are repeated after a certain unit interval, in figure 13 equal to log 4, in the sliderule equal to log 10; and so the distance between any two numbers which are in a given ratio is always the same: also within the unit section the graduations of numbers with equal differences are closer together further on in the scale.

All such functional scales may be constructed also from published tables of the values of the functions. (These tables have usually been calculated otherwise than from graphs of the function.) The values given in the tables should be multiplied by some factor, the **modulus**, so as to make the scale of a convenient size. A rough logarithmic scale may be made quickly by noting that $\log 2 \doteqdot \cdot 3$ and $\log 3 \doteqdot \cdot 475$; from these the graduations for whole numbers save 7 can be marked, and 7 can be interpolated. (*v.* Brodetsky's "Nomography", p. 48.)

If a more accurately graduated scale is required, a chart published with Lipka's "Graphical and Mechanical Computation" (Wiley) will enable this to be to be constructed very easily, provided that the distance between the graduations 1 and 10 is less than 10 inches. Such a chart may, however, be prepared quite easily for any desired scale thus: make a careful copy of a good logarithmic scale; through any convenient point draw straight lines to pass through the graduations of this scale; draw across these radial lines a parallel to the logarithmic scale in the position that gives the length of the required scale.

PERIODIC CURVES

2.3. HARMONIC CURVES: $y = k \sin nx$. Values of trigonometrical functions repeat themselves for every increase of the angle by four right angles. The sine and the cosine thus give the well-known wave curve. The effect of multiplying by some constant factor the variable which represents the angle is simply to compress or extend this wave curve along the axis of this variable. Thus the curve for $y = \sin 2x$ completes an oscillation in an interval of two right angles, that for $y = \sin \frac{1}{2}x$ in an interval of eight right angles. The range of x within which an oscillation is performed gives the *period* of the curve.

These trigonometrical ratios may also be multiplied by constant factors, k, to increase or decrease the range of the oscillation from 1 on either side: this determines the *amplitude* of the curve.

2.31. When several curves of this type, so modified, are combined (**2.16**), interesting **wave-forms** are obtained. The wave repeats itself, of course, after an interval which is the L.C.M. of the periods for the simple curves. In figure 15

are shown by dotted lines the simple curves for the separate terms of $3\sin x - \sin 2x + \frac{1}{2}\cos \frac{1}{2}x$, and by a continuous line the curve got by summing these. One ordinate is drawn to make the procedure of summation clear: $MP = MP_1 + MP_2 + MP_3$; here only MP_2 is a positive ordinate.

It will be noticed that in this case all the curves are symmetrical about the point (180°, 0), and so the compound curve from 180° to 360° is but an inverted repetition of that from 0° to 180°. But the curve just beyond 360° is not the same as that after 0°; repetition does not occur until after 720°, for the longest period, that of $\cos \frac{1}{2}x$, is eight right angles.

A further complexity may be given to the curves by including a term like $\sin (x + 45°)$.

Ex. 1. Point out any defect in figure 15. Complete the curve between 360° and 720°.

Ex. 2. Plot $y = 8 \sin (x° + 80°) - 10 \sin 2x°$.

(Here y is the number of minutes by which the sun is late compared with a perfect clock: sometimes this quantity is negative, *i. e.*, the sun is ahead of the clock. y is called the "equation of time", the quantity which equalises solar time; for the sun is not a uniform time-keeper. The curves which represent the two separate terms are explained in books on astronomy.)

Ex. 3. Take from a newspaper the figures for both the heights and the times of high and low tides. What is the effect of applying the method of **2.18** to each set of these figures? (In the Encyc. Brit. **26** 940 is an interesting figure showing the tides at Bombay: this should suggest to you that series of alternate tides should be examined separately, because of the diurnal inequality.)

One of the big problems of mathematical analysis is to reverse this procedure of compounding curves. The resolving of fractions into partial fractions (**1.1**) is more difficult than the addition of fractions; much more so is the difficulty of analysing a periodic curve into simple harmonic curves compared with that of combining them as above. There is another such problem, *viz.*, that of finding the combination of parabolic curves (such as in **2.16**) which passes through a number of given points. It is not difficult to prove that a curve corresponding to an equation of the nth degree can be made to pass through n points: test this for $n = 2$, 3, etc. Cf. **1.4** f.n.

2.311. A beginning of the analysis of broken curves is made in even elementary books on statistics. In figures of rainfall, temperature, deaths, prices, etc. there are usually sudden changes in successive numbers, and these temporary changes make it difficult to see any general change that may be taking place: cf. **6.52** Ex. 2, etc. In order to make

any such **general trend** clear it is necessary to **smoothe** the curve which represents the numbers. This can be effected by taking the sum of equal groups of successive numbers, and using these sums (or the corresponding averages, called **moving averages**) in place of the original numbers: successive sums are altered by the difference between the number dropped at the beginning of the group and that added at the end, and so the calculation is not difficult.

It is often possible to guess from the general appearance of the graph what the *size of the groups* should be so that departures in opposite directions from the general value in a part of the series may balance one another. If not, we have to proceed by trial of increasing sizes of groups till we find that a certain grouping gives a curve with a tendency that is clear; and we then conclude that by a grouping of this size we have smoothed away the temporary variations which occur in a period the size of the group we have taken.

The next step is to find how the actual figures depart from the values which constitute the general trend. This is done very simply by calculating the differences with proper signs between the actual values and the corresponding smoothed values, and plotting these differences (**deviations** they are sometimes called) against time. Thus will be made clear the nature of the **temporary fluctuations**, whether regular or not: and also the relations between shorter variations in two series, even when there is no relation between the general trends.

Ex. 1. Try grouping the rainfall figures of **7.22** Ex. 2 in successive periods of 1, 2, 3, ...14 years, and consider if any of the grouped series of figures show regularity in their variations.

Ex. 2. Apply the method of moving averages, to the figures in **9.33** Ex. 4 (*e*), (*f*).

2.32. As in **2.22**, **functional scales** may be constructed along the axes for these harmonic curves, though owing to their periodic nature two and more angles may mark the same graduation. Taking the graph of $y = \cos 3x$, (fig. 16) we get along the x axis a scale of cosines of angles which are represented by the distance of the graduation from the origin; and along the y axis a scale of angles, the graduation being at a distance from the origin equal to the cosine of thrice the

angle marked, *i.e.*, the angle marked is what is written $\frac{1}{3}\cos^{-1}y$ (one third of the "inverse cosine" of y) or $\frac{1}{3}$ arc cos y. On the cosine scale the numbers alternately fall and rise in magnitude between 1 and −1 as one proceeds along the scale to the right.

2.33. A very striking presentation of the importance of periodic curves in modern science is the chart that forms the frontispiece to "Phases of Modern Science" (A. and F. Denny), the handbook of the Royal Society to its exhibit at Wembley Exhibition: it shows the diverse nature of the whole known range of **electro-magnetic waves**, *i.e.*, for sixty-two octaves—light waves extend over only one octave! This chart should be copied and hung in every laboratory.

The scale for frequencies is logarithmic, equal intervals, each of two octaves, being marked on the chart. The scale for wave-lengths (being reciprocal to the frequencies, **3,151**, Ex. 2) is also logarithmic, but graduated in the opposite direction. The corresponding diagram in the Dictionary of Applied Physics IV 593, even apart from its smaller size and the remarkably less-developed state of the science which it reveals, is inferior, particularly in the change of unit employed in the wave-length scale. But the indication of comparative lengths given at the foot of the diagram is a very telling illustration of the nature of logarithmic scales. (Our knowledge of these scales tells us that to mark the space occupied by γ rays in this diagram as 6 octaves is a mis-print: an octave is the ratio 2 of frequencies.)

Strictly, however, it is a mistake to mark a length on a logarithmic scale as representing an ordinary distance measured from zero; for the lengths on the logarithmic scale representing all such distances are infinite, the zero graduation being towards infinity in the negative direction (**2.2**): there is no real reason why the line marked 1 cm. in this diagram should terminate on the left where it does. The lines marked 1 cm., 1 km., 5000 kms., taken as the interval between their terminal graduations, represent respectively 0·999,999,999,9 cms.; 0·999,99 km.; 4,999 kms.; but properly they represent merely numbers, the quotients 10^{10}, 10^{5}, and $10^{3.5}$ respectively. Cf. **9.32.**

The chart can with interesting results be compared with the table on pp. 893, 894 of the same volume; though the capricious way in which physicists express wave-lengths in terms of metres, centimetres, millimetres, μ and A is very confusing to ordinary people. The table, though it "shows the entire range of wave lengths of electromagnetic waves which are within the domain of scientific investigation at the present time (1923)," apparently, according to the Wembley chart, does not extend to the wave-lengths used by the British Broadcasting Company!

A similar chart for **sound-waves** is given in this volume of the Dictionary also at p. 699. The compasses of musical instruments are shown by lines of appropriate length. The scale is the musical scale. For frequencies and wave-lengths this is again logarithmic, an octave being represented by a length of about 1·15 cms.

CHAPTER III

INFINITESIMAL CALCULUS

3.11. APPROXIMATIONS: In our study of mathematics probably we have been worried repeatedly by the insistence that has been laid on the very great importance of a correct choice of units and of consistency in their use; but when we think of the world of things round about us we see that great things are done otherwise than by conscious precision such as has been demanded from us. The whole of inanimate nature, rivers, rocks, winds, and stars, "obey," all unconsciously, "laws" which we attempt to express in terms of definite units; and at the opposite pole are poets, some philosophers and others who astonish us also by the brilliance of their intuitions, their "guesses" at truth, though measurement puzzles them. We are not to attempt to measure artists or artistry! But in considering action and reaction even of dead things we seem to need some way of expressing **tendencies,** the effects of the action of very small quantities[1]. Much of what happens to inanimate things, such as we habitually represent by graphs, is a response to what to us is hardly appreciable.

When we looked closely (**1.41**) at particular measurements which we thought to be quite definite, we found that no quantity in nature was expressible with absolute exactness in terms of any unit: there is always a possibility of greater refinement in measurement. Again, even our non-material *ideas* of things are often, if not indeed usually, **approximations.** We are content (and rightly so for ordinary purposes) with a very rough statement as to the facts. Thus to say that a train has an average speed of

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1 So also in economics: cf. pp. 32, 33 of Marshall's "Principles of Economics" [5], a book much of the early part of which is quite easy reading: in his "Economics of Industry", an abridgement of the "Principles", Marshall makes "tendency" practically equivalent to "force", and the argument loses its cogency.

40 miles per hour on a certain journey includes the possibility of its moving at different times during the journey at speeds which we acknowledge as greater or less than that speed; we can define "average speed", but when we describe how that speed is derived from the actual speeds of the train, we find great difficulty. When we attempt to say at what speed the train is moving as it passes a particular point, we find that the only answer we can give is the average speed of the train from that point

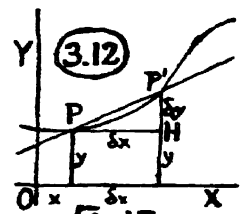

Fig. 17.

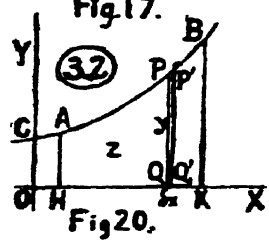

Fig. 20.

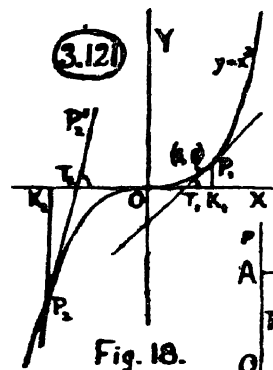

Fig. 18.

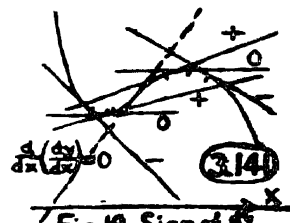

Fig. 19. Sign of $\frac{dy}{dx}$.

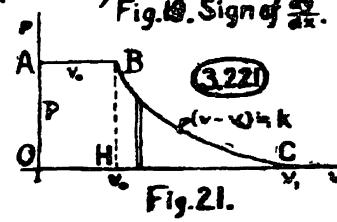

Fig. 21.

to another point not far away; and that answer is indeterminate, for the value obtained differs with the distance at which the second point is taken.

It is worth while working out this idea in detail as it is done in, *e. g.*, Mercer's Calculus, Exercises II.

3.12. THE LIMITING POSITION OF A CHORD: The difficulty discussed in a very incomplete way above is really an advantage to us. We have already used devices (**1.32**) depending on the relations between a tangent-line and chords near it to obtain approximations to the tangent to a continuous curve at a given point. With these devices we are more or less satisfied, for they fit in with our intuitions as to the slope of the curve and its variation from point to point. But what **certainty** is there? The more thoroughly we train hand and eye, the less inclined are we to believe

that they are adequate to a task like this: for two persons, skilled or unskilled, will rarely draw the same line in attempting to draw such a line as a tangent at a specified point to a given curve. And so we are led to search for rules that will enable us to draw such lines; one result of the rules thus found is that we can use tangents to help us to draw curves with speed and certainty.

Taking a point $P\,(x, y)$ on any curve, we can apply the idea of a tangent as a limit and say that the tangent at P is the limit of the endless chord PP' when the neighbouring point P' on the curve comes as close as we like to P. This limiting position we clearly see is quite definite, though the position of the chord itself varies according to the closeness of P' to P.

If we wish to assure ourselves of this definiteness, we consider P' moving up to P from either side along the curve.

Now, if we take the **slope** of a line as the **tangent of the angle** it makes with the x axis, we can readily express the slope of the *chord* PP'. The coordinates of P' differ only very slightly from those of P and may be expressed by $x + \delta x$ and $y + \delta y$, where δ is not a multiplier but merely a sign that a very small **Difference** is made in x or in y. In figure 17, these small changes are obviously PH and HP'. Then

the slope of the chord PP' is $\frac{HP'}{PH} = \frac{\delta y}{\delta x}$, and

the slope of the tangent at P is the limit of $\frac{\delta y}{\delta x}$ as P' comes nearer and nearer to P, *i.e.*, as δx, say[1], becomes smaller and smaller, or as mathematicians write it, $\delta x \longrightarrow 0$.

That is all. We have merely to **apply this** to particular cases, and in doing so to **put it more neatly.**

3.121. Take $y = x^3$: for this we have

$$y + \delta y = (x+\delta x)^3 = x^3 + 3x^2\,\delta x + 3x\,(\delta x)^2 + (\delta x)^3.$$

$$\therefore \quad \delta y = 3x^2\,\delta x + 3x\,(\delta x)^2 + (\delta x)^3.$$

$$\text{or} \quad \frac{\delta y}{\delta x} = 3x^2 + 3x\,\delta x + (\delta x)^2$$

1 We might have regarded δy as tending to vanish; but it is customary to look upon y as depending on x, and it is well on the whole to abide by this custom of regarding δx as leading the way into nothingness!

This is true whatever δx is, even if it be very large : but that has no interest for us at present : that is just ordinary algebra. What happens when $\delta x \longrightarrow 0$? Nothing happens to $3x^2$. But the other two terms, especially $(\delta x)^2$, become smaller and smaller ; and we can make them as small as we like, *i.e.*, negligible, by taking δx small enough. All that is left then is $3x^2$, and this is therefore in this case the limit of $\frac{\delta y}{\delta x}$——it is the slope of the tangent at any point on the curve.

For conciseness, instead of

"the limit of $\frac{\delta y}{\delta x}$ as δx tends to 0" (cf. **3.12**),

or $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$, as mathematicians often put it,

we write Dy, or $\frac{d}{dx} y$, or $\frac{dy}{dx}$ [1]

(where, remember, we mean, not a quotient, but the *limit* of a quotient); and so we have for the simplest cubic curve, $y = x^3$, the compact and universal statement about the slopes of its tangents,

$$\frac{dy}{dx} = 3x^2.$$

Can we test the truth of this? Whatever the value of x, $3x^2$ is always positive; if the tangent of an angle is positive, the angles must be acute; and so all tangent-lines to $y = x^3$ must make acute angles with OX. [2] When x is zero, $3x^2 = 0$.

---

1 Read these latter symbols thus : "d dx of y" or "dy dx", not "dy by dx", which immediately suggests a quotient. This limit, also written sometimes as dy/dx, is usually called the differential coefficient of y with respect to x; because in dealing with small changes in the variables we have the differential formula (**315**), $\delta y = \frac{dy}{dx} \delta x$, where $\frac{d}{dx} y$ occurs as the coefficient of δx. It is better called the *derivative* of y with respect to x. The process symbolised by D is called differentiation.

2 Just as we always take the horizontal arm of the angle to the right along OX, so for simplicity we always take the sloping arm upwards from T. Thus $\angle PTX$, fig. 18, varies only from 0° to 180° for any curve whatever ; but thus also the slope, $\tan \angle PTX$, has the full range of values from $-\infty$ to $+\infty$.

i.e., there is no slope. These tests of the sign of the slope are easily seen to be satisfied in figure 18. By accurate drawing you can verify that the slope measures $3x^2$.

(Such easy questions about signs, etc., you should constantly ask yourself when drawing or sketching graphs.)

Ex. Find the slope of the curves $y=x^2$, $y=x^5$, $y=x^6$, etc. Do you see a regularity in the results you get? Does this suggest a law for the slope of tangents to $y=x^n$?

3.122. We can apply this immediately to the parabolic curves discussed in **2.16.** Just as there we added and subtracted ordinates expressed by the separate terms of an algebraic expression, so here we can add or subtract the differences δy of these ordinates corresponding to a certain small change, δx.

Ex. Find dy/dx when $y=x^2-x^3$. For what values of x is $dy/dx=0$? Draw the curve, choosing a conveniently large scale along the y axis.

3.123. Another obvious extension of this method is to curves like $y=2x^3$. This can be effected by regarding this equation as $y=x^3+x^3$; or preferably by applying the fundamental method once more: thus

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{2(x+\delta x)^3-2x^3}{(x+\delta x)-x}=\lim_{\delta x\to 0}\frac{6x^2\delta x+6x(\delta x)^2+2(\delta x)^3}{\delta x}$$

$$=\lim_{\delta x\to 0}\left[6x^2+6x.\delta x+2\,(\delta x)^2\right]=6x^2=2.3\,x^{3-1}.$$

By these extensions we have made it possible for us to find the slope of the graph which represents any equation of the general form

$$y=ax^n+bx^{n-1}+\ldots\ldots+vx+w.$$

Ex. Test this on some of the examples of **2.16**, **2.17**, and on other such examples.

3.13. STANDARD FORMULÆ: As indicated in the example of **3.121** the general formula for the slope of the tangent at any point of the curve given by the equation $y=x^n$ is nx^{n-1}. This is true for all values of n. We have proved it for only n a positive integer; the proof when n is fractional or negative is lengthy and involves ideas we have

not considered, and so we shall take the result as true in all cases:

$$\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^n = \mathbf{n}\mathbf{x}^{n-1} \quad \text{.....(i)}$$

So also we *assume* the truth of the following formulæ for derivatives[1] :

$$\mathbf{D}a^x = \mathbf{a}^x.\ \mathbf{log}\ \mathbf{a}.\ \mathbf{2{\cdot}3026} \quad \text{.....(ii)}$$

In particular $D10^x = 10^x.\ 2{\cdot}3026$,

$D2^x = 2^x.\ \log 2.2{\cdot}3026 = 0{\cdot}6933.2^x$.

$$\mathbf{D}\log \mathbf{x} = \frac{1}{x}\log e = \frac{\mathbf{{\cdot}4343}}{\mathbf{x}} \quad \text{.....(iii)}$$

e is a constant, irrational as π is, and of value approximately 2·7183.

$$\mathbf{D}\sin\ \mathbf{x}^\circ = \frac{\pi}{180}\cos x^\circ = \mathbf{{\cdot}0175\ \cos\ x^\circ},$$

$$\text{and } D\sin nx^\circ = {\cdot}0175\ n\cos nx^\circ \text{.....(iv)}$$

$$\mathbf{D}\cos\ \mathbf{x}^\circ = \quad = -\mathbf{{\cdot}0175\ \sin\ x^\circ},$$

$$\text{and } D\cos nx = -{\cdot}0175\ n\sin nx^\circ \text{.....(v)}$$

The second particular case under (ii) can be checked by the results of the example in **2.21** : the ratio there evaluated is $\frac{D2^x}{2^x}$

3.131. Those who wish to take a first step into the region of *new ideas* that has been referred to in the beginning of this section may easily connect the second and third (**2.2**) of the above formulæ thus. If a function z, instead of depending directly on x, depends first on a function y which in turn depends on x,[2] then, considering the differentiation of these functions, to a small change δx in x will correspond

---

1 The occurrence of constants (all irrational, it happens) in the formulæ (ii) to (v) for differentiating functions ("transcendental functions" they are called) other than x^n is due to the fact that they are not represented most simply when the scales on both axes are equal. The scales in each case can be so adjusted as to make constant factors unity, *i.e.* they disappear from the formulæ—a comforting instance of the importance of the choice of units (**3.11**)!

2 Another type of example of this imposing of **operation on operation** ("function of a function" it is called in mathematical works) occurs in connection with the taking of logarithms in the second particular case of (ii) above. In finding by logarithms the value of 2·3026 log 2 we have to get log (log 2.).

a small change δy in y with a consequent small change δz in z. Then, by ordinary algebra,

$$\frac{\delta z}{\delta x} = \frac{\delta z}{\delta y}\frac{\delta y}{\delta x}$$

Let all these small quantities become smaller and smaller together. In ordinary cases when there is no sudden change in the functions, each fraction tends to a limit as before, and so

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx},$$

a product of limits of vanishing ratios.[1] In particular, if we put

$$z = 10^y, \text{ then } \frac{d}{dx}(10^y) = \frac{d}{dy}(10^y)\frac{dy}{dx}.$$

Apply this to (iii), $y = \log x$, where $\frac{dy}{dx} = \frac{\cdot 4343}{x}$. The relation may be written $x = 10^y$. Differentiate it in this form: then

$$1 = \frac{d}{dx}(10^y) = \frac{d}{dy}(10^y)\frac{\cdot 4343}{x}$$

$$\therefore \frac{d}{dy}(10^y) = \frac{x}{\cdot 4343} = 10^y . 2\cdot 3026$$

as in formula (ii), where, however, x is the independent variable.

3.132. Formulæ (iv) and (v) are most easily checked by taking x in *radian measure* (**1.9**); then one right angle on the x axis is represented by $\frac{1}{2}\pi$, *i.e.*, 1·571 of the unit on the y scale The formulæ are simply

$$\mathrm{D}\sin x = \cos x,$$

$$\mathrm{D}\cos x = -\sin x \text{; for } x^\circ = \frac{\pi}{180}\, x \text{ radians: and}$$

it can be *verified* that *the slope of one curve is numerically equal to the ordinate of the other.* This is shown at the top of figure 2, where the ordinates corresponding to 10°, 20°,...360° are drawn so that 57° 18′ is represented by the same unit length as is used in the vertical scale. The curve $y = \cos x$ is carried down into the region ruled in squares, where slope of lines can be measured easily. In the figure two tangents

1 Students who are to make a special study of Mathematics must take with caution some of the statements which have been made here. They will learn to appreciate precision of a higher degree than this.

are drawn, at $x=30°$ and at $x=90°$, the points of contact being indicated by arrows. It is easily seen that the inclination of the latter tangent is 135°; to indicate how the slope of the former may be verified to be $-\frac{1}{2}$, two arrows are drawn showing convenient points from which to measure the differences of co-ordinates, *viz.*, -4 for δy and 8 for δx.

The correctness of multiplying the derivate by the factor n in the second generalized formulæ can also be verified graphically (**2.3**); or, more easily from the general formula given in **3.131**: thus, putting z for nx, and using the radian as unit,

$$\frac{d}{dx}\sin nx = \frac{d}{dx}\sin z = \frac{d}{dz}\sin z.\frac{dz}{dx} = \cos z.\frac{d}{dx}(nx) = (\cos nx)n.$$

3.14. MAXIMA AND MINIMA: An interesting and far-reaching application of what we have learned about slopes is to find the **turning points** of functions, *i.e.*, the values of x at which the ordinate changes from increasing to decreasing or from decreasing to increasing as x increases. The former are called maxima, the latter minima. For both of these the obvious condition is that the tangent should be horizontal, *i.e.*, that $dy/dx=0$. Numerous examples of this are given in text books on the Differential Calculus, and only a few typical applications are given here.

Ex. 1. Divide the number 10 into two parts so that (*a*) the sum of the cubes of these parts may be a minimum, (*b*) the product of the two parts may be a maximum.

Ex. 2. A rectangular box with a square top and bottom is to contain 400 cubic feet. The cost per sq. ft. of making the lid, the bottom and the sides is respectively As 4, As 3, As 2. How can the box be made most economically?

Ex. 3. A window is in the form of a rectangle surmounted by an equilateral triangle and has a perimeter of 20 ft. Find its dimensions if it is made so as to admit the largest amount of light possible.

Ex. 4. A beam of length 30ft is to be moved horizontally along a passageway 4ft broad and then turned into another passageway at right angles to the first. What must the breadth of the second passageway be at least so that this may be possible?

Note: There are other ways of determining maxima and minima than by equating dy/dx to zero. Students interested in biology should read the geometrical proof by Roux that the angle θ at which branches leave an artery is given by $\cos\theta = L/L'$, where L, L' are respectively the losses of energy per unit length in the artery and in the branch. (Thompson, "Growth and Form", p. 668.) This is proved also in Feldman's "Biomathematics", p. 173, but by equating a derivative to zero.

3.141. In practical applications it is usually clear whether the value of x for which $dy/dx=0$ refers to a maximum or to a minimum value of y. If, however, we want *a general rule* whereby *to distinguish these cases*, even when what is given us is only an algebraic expression, it is easy to see from a graph that the slopes of the curve at points on either side of the turning point are different in these two cases. For values a little less than that for a maximum the slope is positive, and for values a little greater it is negative; conversely for a minimum value. (Fig. 19).

Thus in the example above, $y=x^2-x^3$, $dy/dx=2x-3x^2$, which vanishes when x is 0 or 2/3. Consider the values of dy/dx when x is $-\frac{1}{8}$ and $+\frac{1}{8}$: they are $-\frac{19}{64}$ and $+\frac{13}{64}$ respectively. There is thus a minimum value of y between $x=-\frac{1}{8}$ and $x=\frac{1}{8}$. Similarly for $x=\frac{1}{2}, \frac{3}{4}$ we have $dy/dx=\frac{1}{4}, -\frac{3}{16}$ respectively, and therefore there is a maximum value between $\frac{1}{2}$ and $\frac{3}{4}$.

3.142. THE SECOND DERIVATIVE: It is usually more convenient to deal with this distinction between maximum and minimum by a further application of the idea of differentiation. Just as,

when the ordinate is increasing, $\frac{d}{dx}(y)$ is positive; so,

when the slope is increasing, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is positive.

Now, at a maximum, does the slope increase as we move along the curve in the direction of x increasing? Before the maximum position it is positive, and beyond it is negative, and so the slope diminishes in passing through this zero value.

Accordingly at a maximum value $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is negative.

Similarly at a minimum value this expression[1] is positive.

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[1] $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ may be written $\frac{d^2y}{dx^2}$ or D^2y, and it has usually been called the second differential coefficient, though "coefficient" is quite inappropriate here. We need not quarrel over the word, however, for the idea is very important: it is yet another example of what we noticed in connection with **3.131**, the repetition of an operation.

This test is very easily applied: in the above example, where $y=x^2-x^3$ $\frac{d^2y}{dx^2}=2-6x$. At $x=0$ this is positive, corresponding to the minimum value; and at $x=\frac{2}{3}$ it is -2, corresponding to a maximum value.

Ex. 1. Construct functions of x, such as those in **2.1** and **2.3**, and find their second derivates. Interpret your results geometrically.

Ex. 2. One more geometrical interpretation is easy, interesting, and useful. When $\frac{d^2y}{dx^2}=0$, what is the corresponding geometrical fact in the figure? It reveals to you a new kind of "tangent". (Cf. the line of dashes in fig. 19.)

3.15. SMALL QUANTITIES: DIFFERENTIALS: Line-graphs are used to show the relation between a great variety of pairs of things, *e.g.*, Boyle's Law for the pressure and volume of a given mass of gas is represented by a hyperbola. Hence what has just been described for curves showing a relation between variables x and y can be applied to the relations between the changes in actual things which are represented by these graphs. In doing this it is specially important to consider small changes (*i.e.*, the **"tendencies"** of **3.11**) in the quantities *e.g.*, the change in the pressure corresponding to a small increase of volume of a gas, or the change in depth in a vessel of given shape (not rectangular) when a given small quantity of liquid is poured in.

Taking $pv=k$ as the formula for Boyle's law we see that a given small change in pressure produces a change in volume that differs under different circumstances. This change can be said to be proportional to the inverse square of the pressure, or alternatively to the square of the volume, or alternatively to the ratio of volume to pressure at the time of the change, as we please. (Note that time is not a variable here.) All this follows from the fact that we can write, since $Dp^{-1}=-p^{-2}$.

$$\frac{dv}{dp}=\frac{d}{dp}\frac{k}{p}=-\frac{k}{p^2}=-\frac{v}{p}=-\frac{v^2}{k}.$$

To express this treatment of small changes a modification of the formula hitherto used is convenient. The fundamental formula was

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{dy}{dx}, \quad \text{or} \quad \frac{\delta y}{\delta x}\to\frac{dy}{dx}, \quad \text{or} \quad \frac{\delta y}{\delta x}\doteq\frac{dy}{dx}.$$

The lefthand side in each case involves a quotient which tends to the limit written on the right. **For practical purposes,** of course, the approximate equation written last

often suffices, and with the understanding that δy and δx are very small ("**differentials**" they may be called[1]) we may write $$\delta y = \frac{dy}{dx}\,\delta x.$$

3.151. From this the standard formula (iii) of **3.13** may be seen to have an important practical meaning: y represents the distance from the graduation 1 to the graduation x on a *logarithmic scale* (**2.22**), and so $y = \log x$. Therefore approximately $\delta y = \frac{dy}{dx}\,\delta x = \frac{1}{x}\,\delta x$. Now δy is the error of measurement we are liable to make habitually owing to the nature of our measuring apparatus, etc., and may be taken to be constant. Therefore $\frac{\delta x}{x}$, *the relative error* in the number read, is *always the same throughout the logarithmic scale.*

Ex. 1. If the length and the breadth of a rectangular metal plate each increase by $\frac{1}{10}$ per cent. per degree as the plate is heated, show that the area increases by $\frac{1}{5}$ per cent. per degree.

Ex. 2. In the theory of light a simple relation connects the wave-length λ with the frequency (or wave number) ν of a particular colour of light, *viz.*, $\lambda\nu = V$, the constant velocity of light, which is 10^8 in the units used. Prove that if at wave-lengths 1200 and 1600 the error in measurement of the wave-length is $< 0{\cdot}1$ the corresponding wave-numbers are $83{,}333 \pm 7$ and $62{,}500 \pm 4$.

Ex. 3. If the law for air under pressure is not Boyle's Law, but $pv^{1{\cdot}5} = k$ show that when the volume is 30 units the relative change in the pressure per unit change of volume is 5 per cent.

Ex. 4. A beam of length l fixed into a wall at one end has a deflection y from the horizontal given by $y = k(\frac{1}{2}\,l^2x^2 - \frac{1}{3}\,lx^3 + \frac{1}{12}x^4)$ at a distance x from the wall. What relative changes are there in the deflection per unit increase in x at the middle of the beam, and at its free end?

3.152. It is often useful[2] to consider the ratio of the rate of change to the size of the changing quantity, *i.e.*, $\frac{1}{y}\frac{dy}{dx}$. This quantity is represented in figure 18 by the *reciprocal of TK, the sub-tangent;* for $TK = KP \Big/ \frac{KP}{TK} = y \Big/ \frac{dy}{dx}$.

1 Properly, the differential dy is the increase corresponding to δx of the ordinate of a point on the *tangent* at P in figure 17, and we can write the *exact* statement $$dy = \frac{dy}{dx}\,\delta x.$$

2 Cf. Marshall's "Principles of Economics", p. 110: also **9.3, 9.46.**

The case when this ratio is constant has been considered in **2.21.** For small quantities we can use the notation of differentials, and convert the *proportional rate of change* $\frac{dy}{dx} / y$ into the proportional change $\delta y/y$ for an interval δx, which may be taken as unity——one year, one foot, etc.

INTEGRAL CALCULUS

3.2. AREA UNDER A CURVE: Besides the slope of a tangent (corresponding to a rate or any other suitable relation between two quantities) there can be obtained another very important geometrical interpretation of the method of reckoning by infinitesimal (*i.e.*, "small") quantities and taking the limit of these when they tend to vanish. In this case instead of evaluating the ratio of two differences, we consider the sum of a great number of differences, *i.e.*, we **integrate** these elemental quantities.

Take a portion AB of *any* curve (fig. 20) and draw ordinates HA, KB through the ends of this arc. As before, let P, P' be neighbouring points on the arc. Then the ordinates PQ, $P'Q'$ include with the elemental arc PP' and OX an area whose breadth is QQ' or δx. If PP' were a straight line, this would be a trapezium whose area would be $\frac{1}{2}(y+\overline{y+\delta y}).\delta x$, *i.e.*, $y\delta x + \frac{1}{2}\delta x\ \delta y$. Now if P and P' be taken very close together, δx and δy will *both* be *very* small, and the area of the strip $PQQ'P'$ may be taken as $y\delta x$: for a product of small quantities, such as $\delta x.\ \delta y$, is obviously small numerically, and negligible, compared with one of these quantities.

This elemental area, $PQQ'P'$, is very small and may be called δz, where z is the whole area $AHKB$; and so we have $\delta z = PQQ'P' = y.\delta x$, or, more accurately, $\delta z \doteqdot y.\ \delta x$, or $\frac{\delta z}{\delta x} \doteqdot y$. This, when there are no breaks in the curve, we can carry to the limit when $x \to 0$, *i.e.*, when P' gets as close to P as we wish; and so we write

$$\frac{dz}{dx} = y,$$

which, being interpreted, means that *the ordinate of any point on a curve is* with respect to the abscissa *the derivative of the area between that curve, OX and any two ordinates.*

3.211. This is a very simple fact, though there is much more we might say about it; and it gives us a new freedom. Apply it to the simplest curve we know, $y=x^2$, *i.e.*, $\frac{dz}{dx}=x^2$. This shows us the new process that is involved in working this out: we have to find what expression z will give us x^2 when we write its derivative, *i. e.*, we have to reverse the process we call differentiation. This **reversed operation** is not always easy, but in this case it is simple. The required function must have an index 3; and $\frac{d}{dx}x^3=3x^2$, therefore the x^3 must be multiplied by $\frac{1}{3}$; and thus we get $z=\frac{1}{3}x^3$, *i.e.*, the area between the parabola $y=x^2$, the x axis and any ordinate at a distance x from the origin is $\frac{1}{3}x.x^2$, *i.e.*, $\frac{1}{3}xy$.

This is at once seen to be consistent with simpler cases; for the corresponding area for $y=x^0$ is $\frac{1}{1}xy$, and for $y=x$ is $\frac{1}{2}xy$; the result for $y=x^2$ should be verified also for such values as $x=0\cdot6$, $1\cdot3$ by counting squares on a graph accurately drawn on graph paper.

The rule for any curve $y=x^n$ is easy to deduce: the integral of x^n is at once seen to be $\frac{1}{n+1}x^{n+1}$, and this we write $\int x^n\,dx=\frac{1}{n+1}x^{n+1}$

$\left(=\frac{1}{n+1}xy,\text{ if so we require to write it}\right)$. Here $\int$ is just an old-fashioned way of writing S, which stands for "Sum", *i.e.*, the lefthand side really means $S\,(x^n.\delta x)$ when δx is taken very small, and here x^n is y in the graph.

3.212. DEFINITE INTEGRALS. We have really been taking for granted in writing this formula that the lefthand ordinate is along the y axis, and for our purpose that would do. But it is easy to win greater freedom and to find the area between *any* two ordinates by taking this as the **difference** of the areas from the ordinate on the y axis to the righthand and to the lefthand ordinate respectively, *i.e.*, in figure 20,

$$HKBA=OKBC-OHAC.$$

If the abscissæ for A, B are a, b respectively we can show the position of the initial and the final ordinates by writing this as

$$\int_a^b y dx = \int_o^b y dx - \int_o^a y dx;$$

e.g., $\int_1^2 x^2 dx$ is the difference between the values of $\frac{1}{3}x^3$ when x is 2 and when x is 1; or, more briefly, $\int_1^2 x^2 dx = \frac{1}{3}x^3\Big]_1^2 = \frac{1}{3}(8-1) = 2\frac{1}{3}$. This should be verified as in the last paragraph by counting squares, and when you feel confident about the result you will have little difficulty in applying this to the other cases.

(The attention that must be paid to signs when part of the curve is to the left of the y-axis will be obvious, or can be learned from a book on the Integral Calculus.)

3.213. It is not intended to attempt further applications of this idea here, but it may be noted that $\int y dx$ can have other significations than mere area. If, as explained more fully in **3.231**, y represents a force acting through a distance x, the sum of the products $y.\delta x$ is the work done by this force, which may or may not vary (cf. p. 3, VI vi). If y stands for the velocity of a body at an instant t, $\int y dt$ is the space traversed by the body between two times which can easily be defined. If x is the rent paid by a number of people y, the integral is the whole sum paid in rents within certain classes of the people. (cf. also **7.3, 9.45,** etc.).

Ex. Redraw the *survivor* diagram (figure 10) as a graph showing the number of deaths per year of age. Thus

for India, between ages 5 & 10, no. of deaths for each year is $\frac{1}{5}(55{,}000 - 50{,}000) = 1000$

„ „ „ „ 10 & 15, „ „ „ „ is $\frac{1}{5}(50{,}000 - 47{,}000) = 600$

„ Italy, „ „ 25 & 30, „ „ „ „ is $\frac{1}{5}(66{,}500 - 64{,}000) = 500$

Plot these numbers of deaths per year of age, taking for abscissæ the middle values of the corresponding intervals. (By this method you will not be able to distinguish clearly between the European countries, as is done in the diagram, drawn in "Medical Biometry", p. 185, from the original figures: but the trend of the curves will be clear.) Thus differences, represented by tangents, are converted into areas. Prove by theoretical considerations, or by counting squares, that the total area under each of these *mortality curves* is the same.

3.22. STANDARD INTEGRALS. To learn to integrate involves learning by heart a set of elementary standard

integrals, which can be combined in dealing with the more complicated functions——practically a new alphabet. We note here the formulæ which are the converse of those in **3.13**, the scales along both axes being the same so as to give the usual meaning to "area."

(i) $$\int x^n\,dx = \frac{1}{n+1}x^{n+1}$$

(ii) $$\int a^x dx = \frac{\cdot 4343}{\log a}a^x.$$

In particular $$\int 2^x dx = \frac{\cdot 4343}{\cdot 3010}2^x = 1\cdot 443.2^x$$

(iii) $$\int \frac{1}{x}dx = 2\cdot 3026 \log x.$$

(iv) $$\int \cos x^\circ dx = 57\cdot 30 \sin x^\circ$$

(v) $$\int \sin x^\circ dx = -57\cdot 30 \cos x^\circ,$$

Note that in (i) n cannot be -1; hence the necessity for (iii).

The particular case of (ii) can be verified by counting squares on the graph of $y=2^x$, used in **2.21** Ex.

(iii) is an instance of an area that extends to infinity in the direction of the y axis and yet is finite when x is finite. Something like this we have seen already in the finite sum of an infinite geometrical progression. (cf. p. 3, VI iii).

The diagrams for (iv) and (v) are very long, the length of one loop being 180 times its height. As already explained in **3.132** the use of the radian instead of the degree as unit of angular measure gives a more manageable figure, and simpler formulæ, *viz.*,

$$\int \cos x\,dx = \sin x, \qquad \int \sin x\,dx = -\cos x.$$

The former of these can be tested on the lower part of the curve $y=\cos x$ shown in figure 2: areas can be counted

there. The area of 100 small squares is the unit area.[1] To verify that, say, $\int_{0°}^{30°} \cos x\, dx = \sin x \Big]_{0°}^{30°} = \sin 30° = \frac{1}{2}$ we count the squares between the x axis and the curve, and betwen the ordinates for 150° and 180° marked in the figure; for this part of the curve is identical in shape and area with that between 0° and 30° (cf. **2.31**). Taking the ordinate for 150° as continuous with the sixth vertical line from the left of the large square below, we count 43 squares and 15 half-squares (p. 2 III) in the stated area. We get thus a value 0·505 for the area, and this is in sufficiently good agreement with theory, considering the nature of our appliances and the approximations with which we have been content. Similarly we can verify the formula between any pair of ordinates we choose.

3.221. THE INDICATOR DIAGRAM. We shall use later (**5.5**) the elementary facts about the expansion of steam in the cylinder of an engine, and it may be well for us to consider these now in order to illustrate formula (iii) above. Steam from a boiler at pressure P is admitted into a cylinder until it occupies a volume v_0 (fig. 21): during this time it pushes the piston along, exerting a steady pressure P, and does work which is represented by the product $P.AB$, or the rectangle $OHBA$. Then, when the volume of the steam is v_0, the supply of steam is cut off, and we have to consider a fixed mass of gas in the cylinder. We take this as expanding according to Boyle's Law, $pv = k$, the pressure falling till it reaches a value which we can take as zero. The state of the steam during this expansion is represented by BC which is an hyperbola. To find the area between this curve and the v axis we have to sum strips of height p or k/v and breadth δv. Then by (iii) $\int p dv = k \int \frac{dv}{v} = 2{\cdot}3026k \log v$. This summation is for a change of volume from v_0 to v_1,

1 The red lines which marked the boundary of the larger squares, the right angle ordinates, etc. on the blackboard did not show up well in the original photograph; and so to make the figure clear they were marked as accurately as possible in black on the plate reproduced. The ordinate bounding the area here described have also been touched up in the photograph.

and therefore its precise value is $2{\cdot}3026k\log(v_1/v_0)$, and the total work done is

$$Pv_0 + 2{\cdot}3026(Pv_0)\log\frac{v_1}{v_0} = P\frac{v_1}{r}(1+2{\cdot}3026\ \log r)$$

where r is the **expansion ratio**, v_1/v_0. There are corrections to be made in this formula for the *back pressure* (which is that of the air if there be no condenser), and for *clearance* (the amount of the cylinder not traversed by the piston—"corners" is the significant word used on the back of the slide rule described in **4.41**), etc. But these do not affect the essentials in the above formula.

(For those who wish to know further details about this diagram, which can be used to show how perfectly an engine is actually working, an elementary statement is easily accessible in the Encyclopædia Britannica[11] **13** 138.)

Ex. 1. Explain how by measurements of the area under $xy=1$ a rough table of logarithms could be constructed.

Ex. 2. Show that $\int\left(\frac{1}{y}\frac{dy}{dx}\right)dx=\int\frac{1}{y}\,dy$, and hence how proportional rates of change (**3.152**) expressed as functions of x may be converted into formulæ which give the corresponding ordinate in terms of x.

This is the converse of **3.152**. (Cf. the first two of Perry's rules, **1.33**, and **9.46**. In many books on the calculus, e.g., Griffin's "Mathematical Analysis", p. 264, Ex. 16, instances of this in physics, etc. are given. A modified case of this conversion of formulæ is of great importance in chemistry, etc., in dealing with what is called *mass action*; cf. Feldman's "Biomathematics", p. 216; etc.: if $\frac{dx}{dt}=k\ (a-x)$, then $a-x=Ce^{kt}$.)

CHAPTER IV

SLIDE RULES

4.1. THE LOGARITHMIC SLIDE RULE: Two *ordinary scales* can be used to perform mechanically addition and subtraction of ordinary quantities. Thus, if the zero of one foot-rule is set to 3·73 inches on another foot-rule laid alongside it, then 6·05 inches on this second scale will be found opposite 9·78 inches on the first; and also the sum of 3·73 and other numbers on the second rule may be read off on the first rule with this one setting. Similarly for subtraction. By sliding the second rule along the first any other numbers shown on the scales may be added. In an exactly analogous way the lengths on *logarithmic scales* (**2.1**) may be added or subtracted, and thus the multiplication and division of the numbers marked on these lengths (which are really the logarithms of these numbers) is effected.

The logarithmic slide rule is an instrument used by engineers and others to read off products, quotients, etc. without actual calculation. At first sight the possibility of getting products and quotients in such a mechanical way is astonishing, and the wonders of the slide rule as usually constructed are indeed many. But once the possibility of constructing scales, other than uniform (**2.22**), is realised, there is little difficulty in understanding how these may be used to give such results.

4.11. THE UNIT RANGE: Just as for ordinary purposes all that is required in a logarithmic table is the logarithms of numbers throughout one range of the decimal scale, *i.e.*, from 1000 to 9999, say, so the graduations on a slide rule need run only from 1 to 10: the positions of decimal points are settled by commonsense considerations, as characteristics are introduced in using logarithms. Slide rules of many different qualities can be purchased, but for the purpose of learning to use the instrument a very inexpensive slide rule may be constructed by cutting from a sheet of

logarithmic graph paper a couple of strips and mounting them on the straight edges of two pieces of cardboard in some convenient way[1]. The numbers corresponding to graduations should be marked in a clear way as on any good scale: too much detail leads in the use of the slide rule to confusion and delay.

It is perhaps better to mount one strip firmly on cardboard first and then with a sharp knife cut through logarithmic strip and cardboard lengthwise: if the cut is straight, it is then easy to bring the graduations into close juxta-position in all positions when the scales are slid along one another. One of the strips should be mounted on a narrow piece of cardboard for the purposes of **4.16**

As indicated above, we shall consider here only the uses of this simplest slide-rule. Once these have been mastered extensions to the use of other slide rules with the unit scale repeated become easy and exhilarating.

4.12. It is scarcely necessary to describe how multiplication and division should be performed, if the description in the preceding paragraph of addition and subtraction by the use of uniform scales has been grasped. It need simply be added that, **if confusion arises** from the complexity of the numbers, the best way to clear one's mind is to *set the slide rule to some simple operation* similar to the operation that causes difficulty, *e.g.*, if ·783 is to be divided by 1519, set the slide-rule first so as to divide 8 by 2 and then it is clear that the slide has to be moved till 1 on it is opposite 1519 on the rule and there need be no hesitation in reading off 516 opposite 783. The usual considerations as to the decimal point then give the result as ·000516.

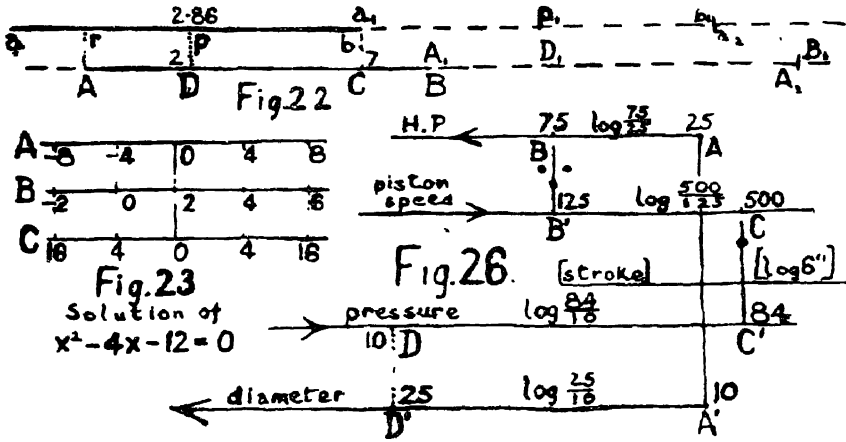
4.13. THE PRINCIPLE: The guiding consideration in all use of this simple slide-rule is that, for any setting, all numbers on the rule and slide opposite one another are in the same ratio. This is easily seen in the case when the slide is set to 2 on the rule, and of course it is true generally that pairs of corresponding numbers at any two positions on the scales give four numbers in **proportion.** The reason may

1 The scales could be constructed from logarithmic tables as mentioned in **2.22**; but this, though a good exercise in accuracy, is too exacting a task for all save those of special aptitude.

be stated thus: representing the numbers on one scale and the other by x and y, we have for any setting the constant interval, $\log x - \log y$, *i.e.*, $\log (x/y)$: therefore, for that setting,

the **ratio x : y** of coincident graduations is **constant.**

This has an immediate application to the converting of quantities in one uniform scale to the corresponding



quantities in another related uniform scale, *i. e.*, to the obtaining of products in which there is a **common factor.**

Ex. 1. 1 kilogram = 2·204 lbs. By setting 1 to 2·204 a whole series of equivalent weights in kgms. and lbs. can be read off at once.

Ex. 2. When the end of the slide is set to 3·14, the circumferences of any number of circles of diameters up to 3·18 units can be read off without moving the slide.

Ex. 3. In an examination the candidates were divided into the following groups according to the marks they had gained: 3, 13, 14, 19, 10, 12. Find the percentage of the total in each of these groups.

Ex. 4, Find pairs of integers whose ratio corresponds most closely to $\sqrt{2} = 1{\cdot}414$. (This is an example of what may be done easily by the slide-rule but would be difficult and uncertain otherwise).

So also for $\pi = 3{\cdot}14$, which is illustrated in the setting of the large demonstration slide rule shown at the top of figure 2. The graduations in this instrument were marked by sign-board painters, and are not very accurate; but they show

coincidences at 4·4, 1·4; 6·6, 2·1; 8·2, 2·6; 8·8, 2·8; 9·4, 3. Other coincidences can be got by moving the slide through its own length (**4.14**). On a good slide rule more or less close coincidences will be found at 1010, 322; 1060, 338; 1080, 344; etc.: but the important ones are those which give ratios of small integers, 22 : 7 and 355 : 113.

Note—For convenience special numbers like π are often marked in their positions on the scales. Chemists have slide rules on which the weights of the more common elements and compounds are marked; so for other special purposes special logarithmic slide rules make calculations easier.

In performing division with the aid of the slide-rule this principle should always be used; but it should be carefully noted that it is much easier to have both the given numbers on the same scale instead of setting them to correspond on different scales. Thus $\frac{\cdot 783}{1519}$ may be written $\frac{\cdot 783}{1519} = \frac{x}{1}$ or $\frac{1}{1519} = \frac{x}{\cdot 783}$, and while either arrangement may be used for the setting, it is distinctly preferable to avoid having to set two numbers like 783 and 1519 in the same position.[1]

4.14. THE SCALE UNLIMITED: If you have tried exercises similar to those suggested above, you will have discovered the difficulty in performing such operations as $3 \cdot 87 \times 6 \cdot 16$ or $19 \cdot 91 \div \cdot 732$. And if you have tried 4×6 and $20 \div 7$ (cf **4.12**), you may have overcome your difficulty by finding that the expected result comes by setting the righthand end of the slide instead of the left. The reasonableness of this you can work out thus in the case $20 \div 7$ (fig. 22):

$$\log 2 \cdot 86 = ap = CB + AD = (AC + CB) + AD - AC$$
$$= \log 10 + \log 2 - \log 7.$$

But you should see it also through imagining the scales doubled towards the right, so that the coincidence between D and p is to the right of B, at D_1 and p_1, and then b is the lefthand end a_1, of the scale which gives the result; or, from analogy with the extent of a 4-figure logarithm table, it should be evident that we can pass from the right to the left

1 To read off the result 516 opposite 783 is easier because the attention is given *successively* to these numbers, whereas in setting 783 to 1519 the attention *alternates* between the two numbers.

of the slide (or rule) just as we can pass from the end to the beginning of the table.

Ex. 1. From the formula

$$O' = \frac{504{\cdot}7\ (R - 0{\cdot}707)}{5{\cdot}047\ (R - 0{\cdot}707) + 4{\cdot}686\ (1 - R)},$$

test the accuracy of the accompanying portion of a table taken from the Du Bois' "Metabolism", p. 39 (cf. **9.522**). Are the zeros in the fourth place significant?

R	O'
·81	36·90
2	40·30
3	43·80
4	47·20
5	50·70

Ex. 2. From the formula

$$P' = \frac{448{\cdot}5\ (1 - R)}{4{\cdot}485\ (1 - R) + 5{\cdot}047\ (R - 0{\cdot}801)}$$

construct some part of a table for values of P' corresponding to values of R between 0 801 and 1. (Cf. the scale of Cal_p shown in figure 58: the whole of this scale can be obtained by dividing the computation among sections of a class of students)

R	P'
·86	$\frac{62{\cdot}9}{{\cdot}629 + {\cdot}298} = \frac{62{\cdot}9}{{\cdot}927} = 68$
7	$\frac{\quad}{{\cdot}585 +} = \frac{58{\cdot}5}{\quad} =$
8	$\frac{\quad}{{\cdot}540 +} = \frac{54}{\quad} =$
9	$\frac{\quad}{{\cdot}494 +} = \frac{49{\cdot}4}{\quad} =$
·90	$\frac{\quad}{{\cdot}4485 +} = \frac{44{\cdot}85}{\quad} =$ ·]

[It is well worth while to do this work in pairs, the "**observer**" (cf. **7.211**), manipulating the slide rule and calling out results, the "**recorder**" stating what is required and arranging the work economically in some such way as is indicated here:

Ex. 3. Check the accuracy of the following table (Du Bois, *op. cit.* p. 232) giving experimental diets for diabetic patients: grammes of carbohydrate, fat and protein are denoted by their initial letters; the calories of heat from these are $4C$, $9F$, $4P$; hence is obtained M in the fifth line; and the formula for the last line is

$$\frac{\text{Fatty Acids}}{\text{Glucose}} = \frac{FA}{G} = \frac{0{\cdot}44\ P + 0{\cdot}9\ F}{U + {\cdot}0{\cdot}58\ P + 0{\cdot}1\ F}.$$

The expression in the denominators gives G. Complete the line for FA/G.

Add a line to show P', the percentage of calories derived from the protein in each diet. (Cf. **9.63**).

		Diet I	II	III	IV
	C ...	10	77	60	51
	F ...	84	108	91	135
	P ...	150	30	85	70
Hence					
Total available	G ...	105	105	118	105
Total calories,	M ..	1400	1400	1400	1700
FA/G	...	—	—	—	1·45

(The 1700 of IV is an obvious misprint; the omission of the three values of FA/G seems to remove the essential point of the table: they are 1·35, 1·05, 1·01.

It should be noted in estimating the trustworthiness of this table that there is no need to retain figures in the last place unless the food actually given to the patient is specially analysed (Joslin, *op. cit.* p. 426). Following Du Bois, p. 231, it should be added, to indicate the significance of the table just referred to, that the object in the treatment of diabetes is two-fold, (i) to keep G so low that there is no abnormal waste, and (ii) to adjust FA so that $FA/G < 1{\cdot}5$, in order to avoid ketosis——whatever that may be! Cf. **9.5281.** This has obviously some relation to the oft-repeated phrase "the ketogenic—anti-ketogenic balance for any level of protein metabolism". Puzzle is out with the help of **9.52** ——if you care!)

4.15. RADICALS AND RECIPROCALS: Finding the square root x of a number y may be taken as a special case of division: given the dividend, find the divisor and the quotient which are identical, *i.e.*, using figure 22 for this new purpose (AB being now the slide), if $ap = \log y$ and $AD = \log x$, then for this setting $ar = rp = AD = \log x = \frac{1}{2} \log y$. Here again the use of the righthand end of the slide may be noted: using the same notation (though with AB as rule now), if $pb = CB$, then it is easily shown that $2AC = AB + AD$. Test this by setting, *e.g.*, $2 \log 4 = \log 10 + \log 1{\cdot}6$.

Note that for every setting the numbers on either scale opposite the end of the other are reciprocals. The reason for this is not difficult to see.

Test papers in the use of the slide rule (cf. **7.31** Ex. 3):

I (*a*) $65{\cdot}3 \times 189$ (*b*) $2070/{\cdot}537$ (*c*) $\sqrt{483}$
(*d*) If 1 kgm. = 2·204 lbs., how many kgms, are in 157 lbs.?

II (*a*) $3880 \times 56{\cdot}6$ (*b*) $1{\cdot}354/58{\cdot}3$ (*c*) $\sqrt{5{\cdot}15}$
(*d*) If 1 km. = 0 621 mi., how many kgms. are in 5·5 mi.?

III (*a*) 5070×469 (*b*) $\cdot0393/{\cdot}697$ (*c*) $\sqrt{636}$
(*d*) If 1 ft. = 30·5 cms., how many ft. are in 11·3m.?

IV (*a*) $83{\cdot}6 \times 644$ (*b*) $51{\cdot}9/{\cdot}2223$ (*c*) $\sqrt{4{\cdot}94}$
(*d*) If 1 yd. = 0·9144m., how many yds. are in 22·5m?

V (*a*) $708 \times 26{\cdot}33$ (*b*) $6{\cdot}95 \times 73{\cdot}3$ (*c*) $\sqrt{6{\cdot}73}$
(*d*) If 1m. = 1·0936 yds., how many metres are in 115 yds.?

VI (*a*) $54{\cdot}8 \times {\cdot}0347$ (*b*) $2{\cdot}34/83{\cdot}5$ (*c*) $\sqrt{777}$
(*d*) If 1m. = 3·28 feet, how many metres are in 682 ft.?

4.16. SLIDE INVERTED: If the slide you have made has been pasted on a narrow strip of cardboard, it can be reversed and the relation between its graduations in this position and those of the rule, which increase from left to

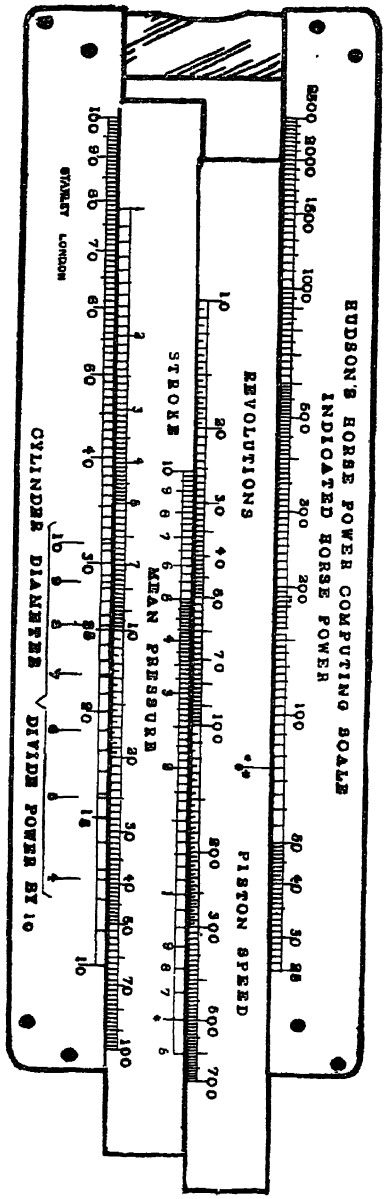


Fig. 24

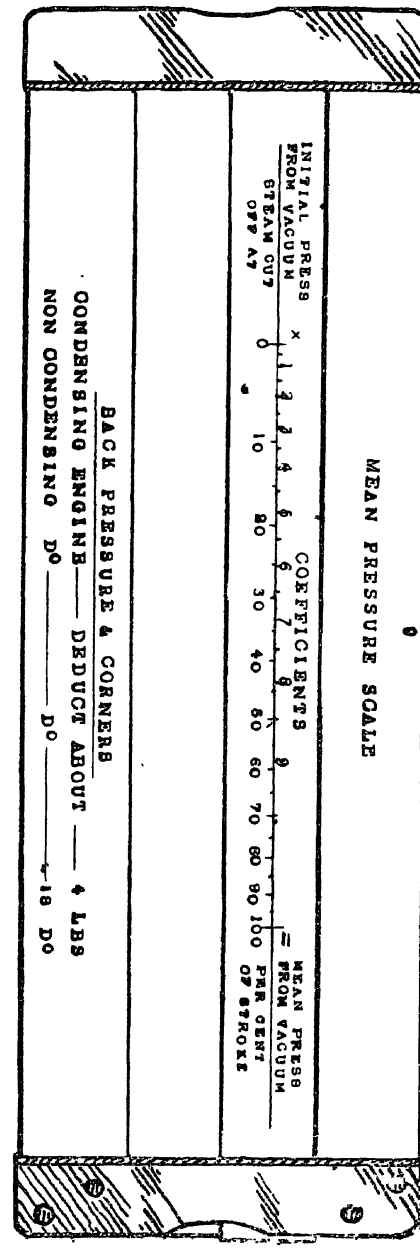


Fig. 25

(These are slightly enlarged views of opposite sides of the same instrument: by accident the back view has been enlarged somewhat more.)

right as before, may be easily investigated. (By the aid of the device called the cursor this can be done on the ordinary four-scale slide rule.) The graduations x on the slide thus placed are distant $\log (10/x)$ from the lefthand end. Thus, if the ends of the scales coincide, graduations x and y opposite one another give (after adjustment of the decimal point) a number and its reciprocal; and for any setting we have the constant difference as in **4.13**

$$\log y - \log (10/x), \textit{ i.e.}, \log (xy)/10.$$

Therefore, for that setting of the inverted logarithmic scale the **product xy** of opposite graduations is **constant.**

These coincident scales may also be regarded as stationary scales for reciprocals : cf. **5.5** f. n.

This arrangement of the scales is convenient when the same number is divided by a series of other numbers. Set one end of the slide opposite the **constant dividend** (which is xy) on the rule : then opposite each divisor on either rule or slide will be found the required quotient.

Ex. 1. Prove that to perform multiplication with the slide inverted the factors should be set opposite one another, and then the product is found on either scale opposite the end of the other. Modify this procedure so as to avoid setting a number against a number other than unity (cf. **4.13**, f.n.).

Ex. 2. State the rule for finding the square root with the slide in the inverted position.

Ex. 3. What with the slide inverted corresponds to the reciprocal property of the end readings in the ordinary position of the slide rule ?

4.2. QUADRATIC SLIDE RULE : Any quadratic equation represented by $x^2+ax+b=0$ can be solved by the use of a uniform scale B (fig. 23) sliding between two fixed scales. (A demonstration slide rule of this type is shown in fig. 2 below the blackboard.) One of the fixed scales A is uniform also, though on half the scale of B; the other C is a scale of squares corresponding to the numbers on B. The expression, $-\frac{1}{2}a \pm \sqrt{\{(\frac{1}{2}a)^2-b\}}$ (p. 3, V), gives the clue to the use of these scales : set zero of B to a on A : opposite this on C is $(\frac{1}{2}a)^2$; from this by calculation subtract b; locate this difference in two places on C; opposite these on B are found the two roots : for the zero of C is at a distance $-\frac{1}{2}a$ from that of B.

In fig. 23 this process is shown for $x^2-4x-12=0$. (So also in fig. 2, where the scale identical with A marked on the upper side of the slide has been inserted merely to illustrate the addition process described in **4.1.**)

4.3. RECIPROCALS SLIDE RULE: The quadratic slide rule is not likely to be of much practical use, for it is unusual to have a large number of quadratic equations to solve, and the use of the rule is not purely mechanical and free from slips in calculation; besides, the equation has to be prepared by division by the coefficient of x^2. Much more likely to be useful would be a rule consisting of two identical scales graduated so that the distance of a mark x from the end is proportional to $1/x$ (**2.22** Ex. 2). The use of such a slide rule where, as in finding the focal lengths of mirrors and lenses, formulae involving nothing but reciprocals occur repeatedly, is obvious. (Cf. also **5.4**)

4.41. THE HORSE-POWER SLIDE RULE: One more example of a special slide rule is given (fig. 24). It illustrates the possibility of combining more than one slide with two scales on the rule. We are not here concerned with the meaning of horse-power, mean pressure, etc.; if information is required about these, it may be found in dictionaries and other reference books. But it is well worth our while to examine how this apparently very complex rule is constructed; for it involves no ideas we have not considered, and from this we can learn how slide rules may be constructed to solve other **special problems.**

A nomogram, as in **5.3** Ex. 3, may easily be constructed to give the same results as the slide rule, but the latter is much more easily carried about by the working engineer: its extra price, compared with that of the nomogram, is not usually a difficulty, but it may more easily get lost! (Cf. for other such considerations, Encyc. Brit. **30** 45*a*).

Instructions given with the slide rule are as follows:

To find Power of Engine:—Set the "mean pressure" on lower slide against the "cylinder diameter", and retaining it in this position bring the "revolutions" on upper-slide, opposite the "stroke" (or the "piston speed" opposite the small arrow)[1] the large arrow will then point to the "power".

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1 The small arrow referred to appears on this slide-rule as an asterisk opposite the graduation 6 inches on the "stroke" scale. The large arrow is more obvious, being represented by three asterisks. Note carefully the feet and inches on the scale for the piston-stroke.

Other instructions are given for finding the size of the cylinder for a given horse-power, *i. e.*, the converse of the above problem, and for other uses of the instrument; but the above is the main problem, and we shall consider only it. In doing so we shall neglect any reference to revolution and stroke, since (how we need not consider) the piston speed is connected with them in an invariable way, and includes the facts as to them. Having thus removed non-essentials, we may *re-state the instructions* for the use of the slide rule, making use of the skeleton-representation of it in fig. 26.

The capitals D, D', C are the points on the respective scales which represent the given values of mean pressure, cylinder diameter, and piston speed. The above instructions then become:

Set D, the mean pressure (p) on the lower slide, against D', the cylinder diameter (d) on the lower rule, and retaining it in this position bring C, the piston speed (s) on the upper slide, opposite C', the small arrow. B' the large arrow, will point to B, the horse-power ($H.P.$) on the upper rule. Then $$AB = A'D' - (C'D - CB').$$

4.42. To interpret this equation we note that all the scales are logarithmic, the metrical scale for each being log3 = 1 inch, save in the "cylinder diameter" scale where it is log3 = 2 inches. In figure 26 are inserted the actual lengths corresponding to the setting photographed in fig. 24. Here we give the general equivalents of the terms in the above equation,

$$\log \frac{H.P.}{25} = 2 \log \frac{d}{10} - \log \frac{84}{p} + \log \frac{s}{125}.$$

(We introduce **the factor 2** in the "cylinder-diameter" term, because a length on the corresponding scale is marked by a graduation which would be shown on half that length on any of the other scales, and therefore it represents the square of that graduation if the other scales are taken as the standard.)

The usual formula employed by engineers is expressed in terms of the area A of the cylinder, and so we change

log $(d/10)^2$ into log $(4A/100\pi)$. Transform now the equation from the logarithmic to the ordinary form, and we get the formula,

$$\text{H.P.} = 25 \times \frac{4\text{A}.7}{100.22} \frac{p}{84} \frac{s}{125} = \frac{\text{s.Ap}}{\mathbf{33{,}000}},$$

which can be easily interpreted by those who know what "work" is (p. 3, VI vi).

4.43. One difficulty remains: why do **the scales run in different directions,** as indicated by the arrows in fig. 26? If we look at the corresponding graduations for horse-power and piston speed as they appear in fig. 24, we see that, as we should expect in accordance with **4.16,** their product is constant. But this is not the clue. We have here an instance (**4.13**) of quotients obtained by division by a constant, 125, represented by the large "arrow", *i.e.*, multiplication by the constant ·008: similarly at the small arrow corresponding to the constant divisor 84. But the difference of direction in the case of the juxtaposed pressure and diameter scales has the effect of giving immediately a product, as in **4.16.** All the scales of products run from right to left.

Reversing the order of consideration of the scales, the effect of the differences of direction in the combination of the scales will become clear. The fundamental direction in which results are recorded is from right to left. The product Ap is recorded implicitly in this direction on the lower slide; to multiply this by s, the scale for s must be (**4.16**) from left to right, the opposite direction to that for Ap; then the scale for Aps is from right to left on the upper slide, and the fraction of this that is required is shown by the large arrow on the horse-power scale, which (**4.13**) is in this same direction.

The whole procedure may be represented thus:

$$\frac{\frac{4A}{100\pi} \times p}{\frac{84}{125} \times s} = H.P.$$

4.44. MODULUS OF A SCALE? In the actual scale photographed in fig. 24 the metrical scale of the mean pressure logarithmic scale is log 3 = 0·995 ins. (The distance between graduations 1 and 81 in the original scale was found to be 3·98

inches.) This is a fraction so close to unity that it makes little difference in practice; it is said that these slide-rules are expected to give results accurate only within 2%. The effect on the theory is that, just as the "diameter" logarithmic term was multiplied by 2, so the "pressure" logarithmic term must be multiplied by 0·995: such a factor is not the scale-modulus of **2.22**; it belongs essentially to the logarithmic formula. The formula for the horsepower that results is

$$H.P. = \frac{s.Ap^{0.995}}{33{,}000}.$$

I have not been able to find out after reasonable enquiry if this is actually the formula for which the slide rule is constructed or not. In the figure of the Hudson's horse-power slide rule shown in the Napier Tercentenary Celebration Handbook, p. 176, the metrical scale of the mean pressure scale is exactly equal to that of the horse-power scale, etc., and it *may* have been that the maker of the slide rule shown in fig. 24 nodded while at his work! If in this case the mean pressure scale had shrunk, the stroke scale would have shrunk also. In any case we should always be prepared to ask questions about our tools——though not be too ready to blame them!

4.45. It is interesting to note how the cylinder diameter scale has been lengthened by inserting at a distance log $\sqrt{10}$ to the right of the graduations for 100, 90, 80,..... graduations for 10, 9, 8,..... with the instruction that, if these are to be used, the number given on the *H. P.* scale is to be divided by 10. This is correct because log $\sqrt{10}$ on the diameter scale is the same length as log 10 on the *H.P.* scale.

CHAPTER V

NOMOGRAMS

5.1. ALIGNMENT NOMOGRAMS: Graphs may be constructed from formulæ so as to give directly values which may also be obtained by the use of slide rules or otherwise. One of the most important ways of dealing with the rather complicated operations represented in such formulæ is to graduate three curves (which in the simplest cases are

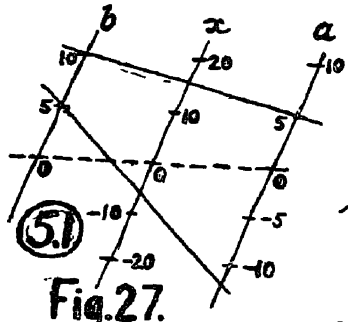

Fig.27.

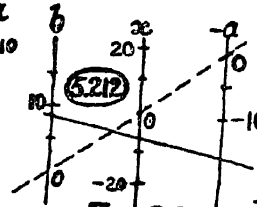

Fig.28.

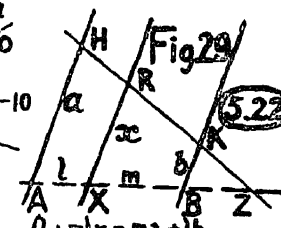

Fig.29.

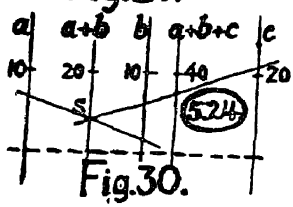

Fig.30.

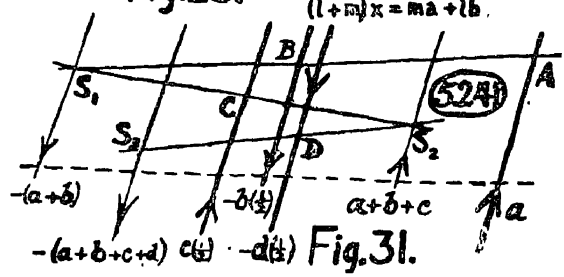

Fig.31.

straight lines) in such a way that by passing a straight line, *e.g.*, a stretched thread, through the graduations for given values on two of these, the graduation on the third, through which this line passes, gives the result of an operation which depends upon the nature of the diagram. Such a diagram is called a *nomogram*, a "law graphed."[1]

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1 More strictly, the **alignment nomogram** described above may be distinguished from **intersection nomograms** which are generalisations of ordinary graphs. Several diagrams of this general type are given later, notably in **5.3** Ex. 6, **9.2**, **9.6**. See also the segmentary nomogram for solving simultaneous equations given in Brodetsky's "Nomography", p. 18 (cf. **5.22** Ex. 3, 4). Engineers use this type of diagram frequently, *e. g.*, along with Ewing's "Steam Engine" are given large drawings to show the relations between such quantities as pressure and total heat of steam under different conditions of temperature, volume and dryness.

The simplest of nomograms is that for **addition,** such as can be performed by two uniform scales sliding along one another. It consists (fig. 27) of three equidistant parallel straight lines, the outermost of which are graduated with identical scales. If the middle line is graduated on a scale half that on the outer lines, the three zeros of the scales being in one line, then obviously a straight line drawn through any two numbers graduated on the outer scales passes through the graduation which marks the sum of these two numbers.

Only a few graduations are shown on the lines in fig. 27; but for this and subsequent figures it should be clearly understood that in nomograms for actual use the graduations can be made as fine as drawing instruments will permit. Here we are concerned only with the principles on which nomograms are constructed.

An easily accessible example of a working nomogram, will be found in the chart, "Table I", for correcting barometer readings as given at the end of Watson's "Practical Physics": the use of the diagram is explained on page 159 of that book, but it is not there called a nomogram. Note carefully how in using this diagram, the trouble of considering anything corresponding to the **proportional parts** in Watson's Table 10 (which is equivalent to the nomogram) is **avoided** Cf. **5.5,** f. n. and Ex. 1. Note also that the **range** of barometer height is distinctly larger, and that of temperature slightly less, in the nomogram than in the table.

5.2. PARALLEL NOMOGRAMS: UNIFORM SCALES: The simple construction for addition given in the preceding paragraph may obviously be generalised and modified in several ways.

5.211. By extending the scales in the negative direction **subtraction** may be performed as the addition of a graduated negative number to one of the original positive numbers; $-11{\cdot}5+5=-6{\cdot}5$ is shown in figure 27.

5.212. Subtraction may also be effected by reversing one of the outer scales so that these two scales are graduated in opposite directions. This is done with the scale a in fig. 28 where the arrangement shown is for finding $x=b-a$, and in particular $8-15=-7$.

5.22. The distance between the graduated lines may be varied so that a line crossing them is divided into segments,

not equal as above, but in a ratio $l : m$. The following easy investigation of similar triangles shows that we thus can effect the addition of **given multiples** of the numbers on the outer scales.

Let A, X, B be the respective zeros of the scales, and let any straight line cut the graduated lines in H, R, K respectively, the points graduated a, x, b respectively, and the line of zeros in Z. Then by a simple construction

$$\frac{AH-XR}{XR-BK} = \frac{AX}{XB} = \frac{l}{m},$$

or

$$(l+m)\ XR = m.\ AH + l.\ BK.$$

This equation may be interpreted as equivalent to

$$x = ma + lb.$$

and then it is obvious that the scale on the inner line is that on the outer lines divided by $l+m$: for we took all the lengths XR, AH and BK to be measured in terms of the same unit. The nomogram we first considered was the special case of this where $l = 1 = m$. The above proof can be modified for all positions of HRK, due attention being paid to signs. Thus *for a given value of* x *all straight lines through the corresponding graduation will cut the* a *and* b *scales in values of these variables such that* $x = ma + lb$.

Ex. 1. Construct a nomogram for the formula $v = u + ft$, where f is a constant -32, 981, or any other number.

Ex. 2. The length s of a circular arc is given approximately by $3s = 8l - L$, where l is the chord of half the arc, and L the chord of the whole arc (Borchardt and Perrot, Trignometry, p. 306). Construct a nomogram to represent this formula.

Ex. 3. Solve for a and b by means of a nomogram such simultaneous equations as $15 = 4a + b$, $-10 = a + 2b$. (Construct parallel equal scales for a and b, and between them appropriate scales like XR for say x and x'; the straight line joining the points $x = 15$, $x' = -10$ will when produced cut the a and b scales at the required values.)

Ex. 4. Construct a nomogram to represent the relations,

$$0{\cdot}024\ M = C + 0{\cdot}41P$$
$$F = 4C + 1{\cdot}4P$$

(These may be regarded as simultaneous equations in C and P and dealt with as in the preceding example. But C, F, and P have the meanings stated in **4.14** Ex. 3, and M denotes the number of food calories required per day. The medical application of the nomogram, which is given in Du Bois' "Basal Metabolism," p. 235, and also in Joslin's "Diabetes Mellitus"[3], p. 461, is to find C and F when P and M are given.)

5.221. With a view to noting the flexibility of this method depending on similarity of triangles, it may be remarked that the equation

$$(l+m)\,x=(l+m)XR=m.AH+l.BK=ma+lb$$

may be used to effect simple addition by making the scales on the a, b and x lines, not equal as in figure 29, but some standard scale divided by m, l and $m+l$ respectively for these lines.[1] This fact makes the converse problem, that of getting the equation from the mere figure, indeterminate, unless the scales are given. Cf. **5.3** Ex. 6.

5.23. A CONSTANT TERM: An obvious modification of figure 29 is to displace the x line along its own length until the graduation at X is c. The effect of this is to make the reading at R greater by this constant, and thus the formula corresponding to this arrangement is

$$x=ma+lb+c$$

Note—If u is a constant in $v=u+ft$, the formula becomes of the type $y=mx$, which leads to another simple type of nomogram, that used in **9.6.**

5.24. If the summation is to extend to **three or more terms** which involve variables, the above process has to be repeated, the terms being taken in pairs in any way that is convenient. As the final result alone is required, it is unnecessary to graduate the lines on which the partial sums are found. Such a line is called a **reference line** or a **dummy axis.** Thus in figure 30, which represents the simplest case, all that we need to know on the $a+b$ line is the position of S, where it is cut by the line joining the graduations on the a and b lines. It will be clear that the scale on the $a+b$ line is half that on the a line; and so also must be the scale on the c line, if the $a+b+c$ line is taken midway between the $a+b$ and the c line. The scale on the $a+b+c$ line will then be $\frac{1}{4}$ of that on the a line. And so on.

Much ingenuity may be spent in arranging such nomograms so as to secure ease and accuracy of reading, and also

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1 i.e. multiplied by l, m and $\frac{lm}{l+m}$, as it is put in Feldman's "Biomathematics," p. 121: so also Lipka, *op. cit.* p. 45. These quantities, or $1/m$, $1/l$, $1/(l+m)$, are the respective scale-moduli; cf. **4.44.**

other advantages. It is held that these are attained more readily if the whole nomogram occupies an area on the paper that is nearly square: the likelihood of oblique intersections, the graduations at which are difficult to read, is thus reduced. Brodetsky, in his "Nomography", page 11, lays it down too that we must as much as possible **avoid** the use of **widely differing units**. This is so; but a consideration of possible errors (**1.34**) shows that there is no disadvantage in having to read off the result x of a sum or

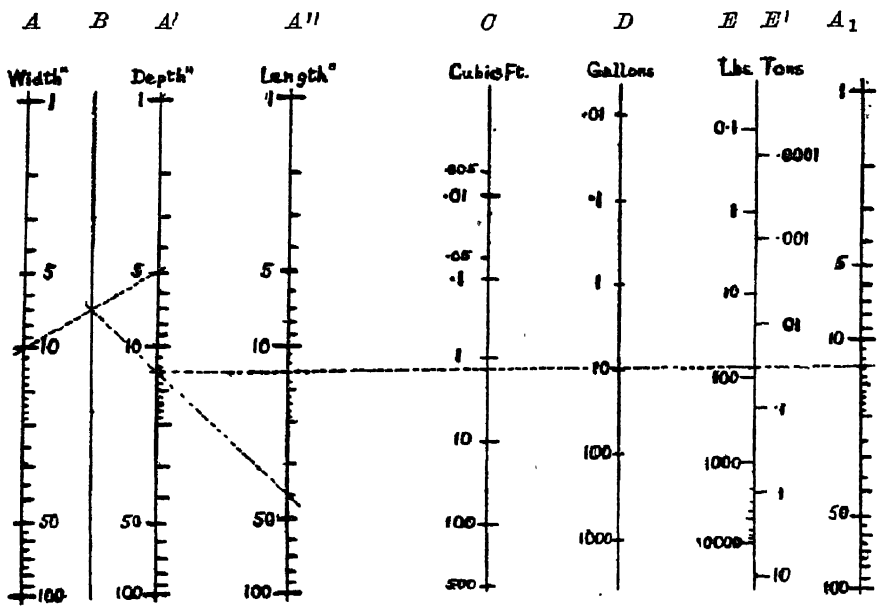

Fig. 32. Chart showing the capacity of a Rectangular Tank. (Modified from the Textile Recorder, 1923, p. 669)

difference of two quantities, a and b, on a scale half those on which the given quantities are marked: the possible error of the result, $a \pm b$, is twice that of the given quantities, and the accuracy that is obtained by reading off a result on a scale equal to that used for either given quantity is illusory. But it is well to note (as Brodetsky emphasises) that the nomogram may be modified so as to bring the line giving the result of addition or subtraction to the outside of the diagram.

5.241. This is done simply by rearranging $x=a+b$ as $-a=b+(-x)$; in the nomogram the corresponding change is that the outer lines are those for b and x graduated on the same scale in opposite directions, while the middle line is that for a graduated on half the scale in the negative direction. Thus a diagram may be built up to determine the sum of any number of terms. In figure 31 the process is shown repeated thrice, as indicated by the arrows. For simplicity the coefficients of the terms are taken as unity; thus there are only two scales in the figure, the scale of a and all the dummy axes, and the scales of b, c,....., half that of the former. This is indicated in the figure by $(\frac{1}{2})$, and the directions in which the graduations are to be marked are shown by the signs attached to the quantities, as well as by the arrows.

To emphasise the method of arrangement the lines on which the terms are graduated are drawn rather heavily. In a working nomogram this should **never** be done, as the accuracy with which graduations can be read is much reduced if a line is thick.

Ex. 1. Re-draw the nomogram for $v=u+ft$, (**5.22** Ex. 1.), so as to bring the line for v to the outside of the diagram.

Ex. 2. By purely statistical methods the following formulæ have been found for M, the heat production for 24 hours of individuals of weight W kgms. height H cms., and age Y years (Du Bois, *op. cit.*, p. 161):

For men $M=66{\cdot}4730+13{\cdot}7516\ W+5{\cdot}0033\ H-6{\cdot}7550\ Y$;

for women $M=655{\cdot}0955+9{\cdot}5634\ W+1{\cdot}8496\ H-4{\cdot}6756\ Y$.

Taking approximate values for the constants, construct a nomogram for each of these formulæ. (Cf. **9.5271**).

5.3. PARALLEL NOMOGRAMS: NON-UNIFORM SCALES: The rules given in the preceding paragraph are comparatively unimportant in their application to the cases so far dealt with; for it is not usually worth while to construct nomograms to effect mere addition. But there is no need to restrict the graduations of lines to uniform scales. Just as in the case of the slide rule, the lengths on logarithmic scales in a nomogram may be combined so as to get the products of the numbers corresponding to the given graduations. The same is also true of other

scales, *e.g.*, squares, square roots, reciprocals; and nomograms can be constructed with ease for formulae like

$$c^2=a^2+b^2,^* \qquad \frac{1}{v}+\frac{1}{u}=\frac{1}{f}.$$

CAPACITY OF A RECTANGULAR TANK: The accompanying figure, 32, is adapted from a Cotton Trades publication. The nomogram consists of the four lines on the left of the figure. It is noteworthy that the three logarithmic scales A, A', A'' which are used are identical. The three lines, C, D, E, to the right of these merely carry scales on any of which the result may be read off according to the form in which it is required. A duplicate A_1 of the logarithmic scale of the nomogram is given on the extreme right so as to facilitate setting parallel lines across the figure. Frequently equivalent scales are marked on either side of one line, and this has been done here with the very similar scales for pounds and tons of water, though in the original diagram they were shown on separate lines (**5.5** f. n.).

It is important to notice that a simplification of the figure has been effected through using the scale A' both to mark the depth of the tank and to record the final product. Yet none of the scales on which this product is read off is marked on A' itself; hence the importance of drawing a horizontal line accurately across the figure.

Having noted these things, the construction of the nomogram is easily unravelled. The sum of the logarithms on A and A' is given on the reference line B on a half-scale (not marked). The sum of twice the half-logarithm on B and

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* A nomogram for this formula is worked out in Hezlet's "Nomography," (Royal Artillery Institution, Woolwich) p. 25: in the figure, however, the graduations 1 for the scales a and b clearly are not inserted accurately—the interval from 1 to 1·5 is shown as greater than that from 1·5 to 2; also some of the other graduations are not consistent. It would have been well to have carried these scales back to the graduation 0; for they are not logarithmic. In graduating the scales it is unnecessary to use Hezlet's general formulæ; the calculation may be arranged very simply thus:

Graduation	1, 2, 3, 4,.....
Distance from zero graduation, a and b scales	1, 4, 9, 16,.....
c scale (cf. **5.1**)	·5, 2, 4·5, 8,.....

the logarithm on A'' is given on A' whose distances from B and A'' are as 1 to 2. The scale of the logarithms for this result is one third that of the original scale, but this again is not marked on the line A': the result is read off on either C, D, E or E', which are all logarithmic scales. The distances between these last lines have of course no significance.

Ex. 1. Redraw on a larger scale fig. 32, and insert more numerous graduations (**2.22**). Note that the line joining the 12″ graduations passes through 62·5 on the E scale: this agrees with the fact you learn in physics, that the weight of 1 c. ft. of water is approximately 62·5 lbs. Check the accuracy of the nomogram by calculating values for each of the scales for a tank of specified dimensions. Draw a similar nomogram for larger tanks.

Ex. 2. In Gregory and Hadley's "Classbook of Physics" p. 560 is given a table of the maximum vapour pressure p (in mm. of mercury) of water at different temperatures t. The formula for these values is

$$\log p = A + \frac{B}{\theta} + C \log \theta, \quad \text{where } \theta = t + 273.$$

(Kaye and Laby "Physical and Chemical Constants," p. 40.)

Show how to construct a nomogram which is equivalent to this formula, and state the modifications that are necessary when negative values are given to some of the above constants thus: [1]

$$\log p = 15{\cdot}24431 - \frac{3623{\cdot}932}{\theta} - 2{\cdot}367233 \log \theta \quad \text{for values of } \theta \text{ from } 15° \text{ to } 270°;$$

$$\log p = 10{\cdot}04087 - \frac{3271{\cdot}245}{\theta} - {\cdot}7020537 \log \theta \quad \text{for values of } \theta \text{ from } 270° \text{ to } 450°.$$

(Kaye and Laby, *op. cit.* p. 41)

Show how the nomograms for these two formulæ may be arranged in one diagram in which the reciprocal scale and the p scale are the same for both formulæ.

Ex. 3. Devise a nomogram for the formula found for the horse-power slide rule in **4.42**.

(Writing the formula as

$$\log H.P. = \log s + 2 \log d + \log p - \text{a constant},$$

there is little difficulty in constructing this nomogram; but a detailed treatment of it, if required, will be found in Lipka's "Graphical and Mechanical Computation", p. 63.)

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1 These formulæ are given in full in order to indicate a limitation to which nomograms are subject. Only a very skilled draughtsman could prepare a diagram in which all the figures of these constants would be significant, and even then separate diagrams would have to be drawn for successive ranges of the values of θ. Cf. the remarks about the superiority of arithmetical methods over graphical methods in the preface to Whittaker and Robinson's "The Calculus of Observations" p. vi, and the incident (already referred to in **4.41**) related in the Encyclopædia Britannica, **30** 45 *a*.

Ex. 4. Construct a nomogram for $\triangle = \frac{1}{2} ab \sin C$.

This may be taken as $\log \triangle = \log a + \log b + (\log \sin C + \log \frac{1}{2})$; then, following the method of **5.241**, we may construct the a logarithmic scale in what we take to be the positive direction, and parallel to it the b logarithmic half scale in the opposite direction. These give us a dummy axis for ab. With values (which are all negative) from a table of logarithmic sines the scale for $\log \sin C$ may be inserted in the positive direction with half-unit compared with the a scale, but displaced a distance log2 along the direction of the b scale. Thus, the final result being to be read off on a scale identical with the a scale we started with (*i.e.* $\triangle = a$ numerically) the zero of the $\log \sin C$ scale (*i.e.*, 90° in the C graduations) must be opposite one graduation 2 on the b scale; also the graduation 6° will be nearly opposite the next graduation 2 on the b scale, for $\log \sin 5° 44' = -1$,

(Those who cannot construct the figure from this description will find it on page 106 of Brodetsky's "Nomography". The figure there might have been reduced to one of three lines if the C graduations had been placed on the same line as the b graduations, one set on each side of the line; and then the area would have been shown on the a line, the scale on it being the same as for the area ab on the dummy axis, though in the opposite direction, as in **5.241**. The double set of graduations on the C scale, each pair totalling 180, is not necessary, though it should be noted: if it is thought advisable the two numbers can be inserted side by side on the same side of the line without risk of confusion : cf. fig. 16.)

Ex 5. Devise a nomogram for $\tan \frac{1}{2}(B - C) = \dfrac{b-c}{b+c} \cot \frac{1}{2}A$.

Here b is taken greater than c; otherwise we should have to deal with the logarithms of negative quantities. The formula is equivalent to

$$\log \tan \tfrac{1}{2}(B - C) + \log \tan \tfrac{1}{2}A = \log \frac{b-c}{b+c},$$

and the same method of representation as was used in the last example may be employed, though the number of terms is less. From a table of logarithmic tangents a scale for $\log \tan \frac{1}{2}(B - C)$ is constructed; and parallel to it, but in the opposite direction and with half the unit, a scale for $\log \tan \frac{1}{2}A$. The unit may be checked by noting that $\log \tan 45° = 0$, $\log \tan 84°17' = 1$; and so the intervals between the graduations 90° and 168° 34′ (or 11° 26′, for the scales are symmetrical about the graduation 90°) on the $B - C$ and the A scales will be 1 and $\frac{1}{2}$ respectively. The scale for $\log \dfrac{b-c}{b+c}$ must be calculated; it is most convenient to mark its graduations with values of the ratio b/c which is easily calculated. The position of a particular graduation on the b/c scale may be fixed by considering, say, $B - C = 60° = A$; then $(b-c)/(b+c) = \frac{1}{3}$ or $b/c = 2$. If the b/c scale is to be equidistant with the $B - C$ scale from the A scale, then the same unit is to be used for $\log \dfrac{b-c}{b+c}$ as for $\log \tan \frac{1}{2}(B - C)$, but the graduations are to be in the reverse direction. The arrangement of the nomogram is now determined, and all that remains is to insert the appropriate graduations from calculated values of $\log \dfrac{b-c}{b+c}$; *e. g.*,

b/c	19	9	5	3	2	1·5	1·25
$\frac{b-c}{b+c}$	$\frac{18}{20}$	$\frac{8}{10}$	$\frac{4}{6}$	$\frac{2}{4}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{9}$
$\log\frac{b-c}{b+c}$	$\bar{1}\cdot9542$	$\bar{1}\cdot9031$	$\underline{\bar{1}\cdot8240}$	$\bar{1}\cdot6990$	$\underline{\bar{1}\cdot5228}$	$\bar{1}\cdot3010$	$\underline{\bar{1}\cdot0457}$

(This again is a description of a nomogram to be found in Brodetsky's "Nomography", page 107. It is given as a further test of the student's grasp of the principles that have been described. The figure given by Brodetsky is not quite accurate, for the line joining the two graduations 60° passes the graduation 1·95 for b/c. A more thorough test of the accuracy of the figure may be made by considering the three logarithms underlined in the table above. Measurement gives the distance between graduations 5, 2 as 2·36 cms., between 2, 1·25

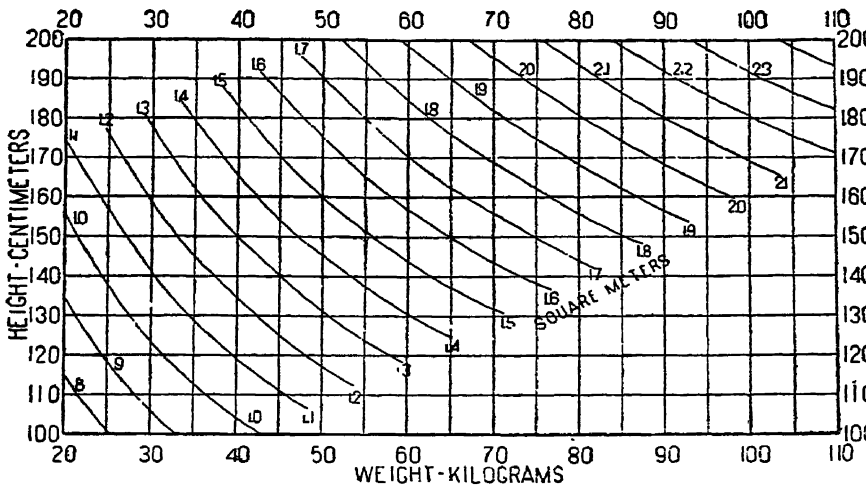

Fig. 33. Chart for determining surface area of man's body.

as 4·13 cms.: these give, instead of a constant logarithmic scale, 7·87 cms. and 8·66 cms. as the scale per unit difference of logarithm in the two parts of the scale. This nomogram applied to a set of examples (XX in Borchardt and Perrot's "Trigonometry") was found to be inapplicable to five out of twelve because the range for b/c did not extend below 1·25. For the remaining seven the average error in the value of $B-C$ was 1°35′ with a maximum error of 3°14′. Other arrangements for the nomogram can of course be devised.)

Ex. 6. Construct a nomogram to determine the surface area A sq. metres of the human body of weight W kgms. and height H cms. from Du Bois' equation,

$$A = 71{\cdot}84\ W^{0{\cdot}425}\ H^{0{\cdot}725}.$$

(This is a straightforward piece of work, but those who need help or who wish to check their work will find a nomogram for this formula given in Pearl's "Medical Biometry" (Saunders) p. 134; also in Feldman's "Biomathematics"

(Griffin) p. 124, where only C. G. S. units are given: these nomograms as reproduced, though from the same source, do not give quite the same results. Feldman gives a description of the construction of the nomogram.

It is interesting to compare this diagram with two others for the same formula given in Du Bois' "Basal Metabolism" (Lea and Febiger) pp. 142, 3. The former is an intersection nomogram (**5.1** f.n.), reproduced here much reduced as figure 33; the latter, another alignment nomogram, which is marked "Copyright, 1920"; but the fact that it forms part of a later chart (reproduced as figure 1, and described in **5.31**) which has not been copyrighted indicates how rapidly the nature of such figures has become common knowledge. This alignment nomogram

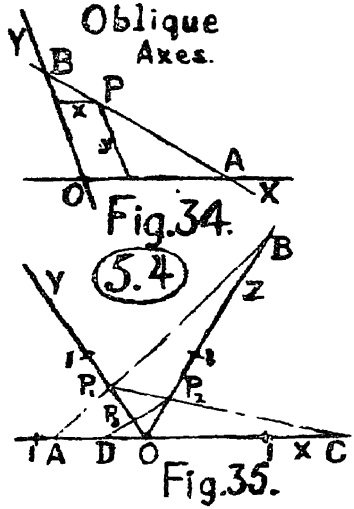

Fig. 34.

Fig. 35.

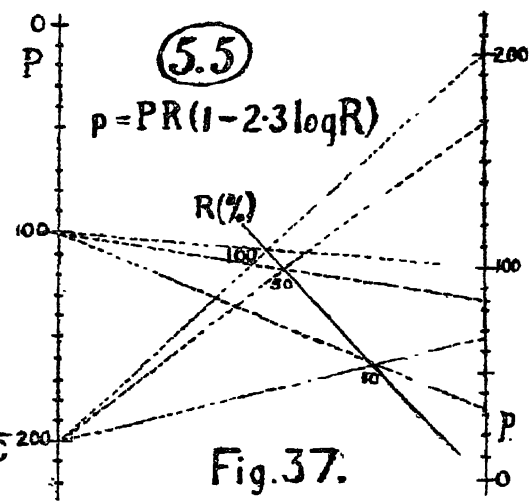

Fig. 37.

differs slightly in plan from that in "Medical Biometry," and it gives a better range of values, but it is overloaded with figures; their readings also do not always correspond, *e. g.*, $W=140$ lbs., $H=5$ ft. 6 inches give respectively 1·70 and 1·71 square metres: the intersection nomogram gives 1·72 or 1·73. Cf. also **5.61**.

A fourth nomogram, and the best of them in the selection of ranges for variables, is that given by Wilson and Wilson in the "Lancet" **199** 1044. It may be re-drawn from the following specification, and its construction investigated: the H, A, W graduations for 150, 1·3, 40 are on a line perpendicular to the scales, with distances 3·93 cms., 2·07 cms. between them; and the distances in centimetres of graduations from these graduations along their respective scales are

for H 160, 170, 180, 190, 200; 2·66, 5·14, 7·52, 9·77, 11·89 respectively;

for A 1·4, 1·6, 1·8, 2, 2·2, 2·4; 0·96, 4·08, 6·42, 8·52, 10·42, 12·15;

for W 50, 60, 70, 80, 90, 100; 2·89, 5·25, 7·23, 8·98, 10·52, 11·88.)

Ex. 7. Construct nomograms for the following surface-area formulæ for the human body (from "Basal Metabolism", p. 144): the average error for each

formula, and in some cases the maximum error (in the case of fully-grown bodies), are noted for interest.; nots also the indices.

				Av. error	Max. error
(i)	$A = 167{\cdot}2$	$W^{\cdot5}$	$H^{\cdot5}$	... 2·2 per cent.	−5·8 per cent.
(ii)	$A = 25{\cdot}6$	$W^{\cdot333}$	H	... 3·3 per cent.	
(iii)	$A = 12{\cdot}312$	$W^{\cdot667}$	(Meeh 1879)	15 per cent.	(Cf. **5.23.** Note)
Cf. for	$A = 71{\cdot}84$	$W^{\cdot425}$	$H^{\cdot725}$	... 1·7 per cent.	2 per cent.

Note—Other formulae for finding metabolism are discussed in the *Lancet* **199** 290, and these may similarly be reduced to nomograms, if desired: a quantity W called "*the theoretical weight*" (less likely to be altered by illness than the actual weight) is obtained from the trunk length and chest measurement; and this is directly connected with calorie requirements M and age Y by

$W^{\cdot5}/(MY^{\cdot333}) =$ a constant, *viz.*, ·1015 for men, ·1127 for women.

5.31 BASAL METABOLISM. A nomogram corresponding to the **multiple type** of **5.24** is reproduced as figure 1:[1] counting from the right, the first, second and fourth lines form the nomogram referred to in **5.3** Ex. 6 as marked "copyright". The purpose of this part of the nomogram is to obtain the body area A. This is proportional to the metabolism M of the body. The factors by which A must be multiplied to give M vary, however, and all such factors N are given on the lefthand line. This is marked on a logarithmic scale for N, since N is simply $M/24A$; and so this part of the nomogram is of a simpler type than the other.

Along this N scale are marked two scales of the ages of men and women at which N has been found experimentally to have the values indicated. The age scales are not regular. Note that for boys of 12 to 13 years there is a check in the fall of the standard metabolic rate; so also, though less markedly, for men of 50 to 59, and for women of 40 to 49. Cf. **2.0** Ex. 2, 4: in figure 8 the comparison is with weight, not area.

Ex. Fix a point on the N scale, and note that all straight lines through it cut the M and A scales at graduations which are in the ratio, 24 times the N graduation.

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1 This is given on the page preceding that referred to in each book cited in **5.22** Ex. 4.

5.4. INTERSECTING SCALES. The lines on which graduations are marked in a nomogram need not be parallel, or indeed rectilinear. The theory of such nomograms is usually, however, rather more difficult than in the cases we have considered, and only one or two simpler cases will be dealt with here.

The equation to a straight line $\frac{x}{a}+\frac{y}{b}=1$ gives a very simple nomogram for reciprocals. If $x=y$, *then* $\frac{1}{a}+\frac{1}{b}=\frac{1}{x}$. The three lines involved in the corresponding figure are obviously the axes and the bisector of the angle between the axes, and the scale on the bisector is $\sqrt{2}$ times that on the axes.

Hitherto for any purpose we have not considered axes the angle between which is not 90°. But it is easy to see that the equation $\frac{x}{a}+\frac{y}{b}=1$ depends solely on similar triangles; for (fig. 34) $\frac{x}{a}+\frac{y}{b}=\frac{BP}{BA}+\frac{AP}{AB}=1$. The equation is therefore true whatever the value of the angle between the **oblique axes,** OX and OY. If this angle is made 120°, the **scales** on the two axes and the bisector of the angle between them become **equal.** This makes it possible to get a simple diagram on which the sum of any number of reciprocals can be quickly read off. In figure 35 the quantities $\frac{5}{6}$, $2\frac{1}{2}$, $1\frac{2}{3}$, $\frac{1}{3}$,...whose reciprocals are added are represented by OA, OB, OC, OD, OE,.....The successive sums are marked on the sloping lines OY, OZ alternately. The points are noted with great ease if a straight line ruled on a transparent celluloid sheet is used to connect successive points; or by two workers each using a fine thread.

Ex. 1. Why do successive Ps come nearer to O? Does this fact affect the worth of the nomogram?

Ex. 2. Construct nomograms of this type for

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}. \qquad \frac{1}{\sin A}+\frac{1}{\sin B}=\frac{1}{\sin C}.$$

5.5. EMPIRICAL NOMOGRAMS: Most important is it to note that in practice it is not necessary to know the

theoretical basis of a nomogram before being able to construct it. If either from a formula or from experiment we can get a sufficient number of relations between the given variables a and b and the unknown variable x, then it is usually possible to get from *the intersections of lines joining pairs of values of a and b which give the same value for x*, a series of graduated points for x which lie on a curve. From these points the x scale can be completed by interpolation.

Many examples of this mode of procedure can be found in textbooks or articles. We shall consider here how to devise an alignment nomogram to serve the purpose effected by the scales[1] on the back of the horse-power slide rule, figure 25. In **3.221** we found the work done by steam in the cylinder to be $Pv_1(1+2\cdot3\log r)/r$. It follows that the **mean pressure** p during the increase of the volume of the steam in the cylinder to a value v_1 is $P(1+2\cdot3\log r)/r$, where constants relating to condenser pressure, etc., are omitted for simplicity. r here is the expansion ratio v_1/v_0: this we can replace by its reciprocal, R, the ratio v_0/v_1 at which the steam is cut off, and the formula for the mean pressure becomes

$$p = PR(1-2\cdot3\log R)$$

This is the formula that is required by the scales on the back of the horse-power slide rule, *e.g.* if $R=\frac{1}{10}$ or 10%, then $\frac{1}{10}(1+2\cdot3\log 10)=\cdot33$, which is the corresponding reading on the scale of coefficients.[2]

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1 These are called *fixed* or **stationary scales**, or simply *double graduations*, and are sometimes classed as nomograms. An example has been noted in passing in the nomogram for the capacity of a tank, **5.3**. Other examples may be found in the Dictionary of Applied Physics, III 635, or, more fully, in Lipka's "Graphical and Mechanical Computation", p. 5. The graphical table of logarithms illustrated in figure 36 is another example of stationary scales: note that here, as in **5.1** there is no need to trouble with proportional parts to get the fifth decimal place; all that is needed is skill in reading a scale (**1.31**). Another example is the relation between the fixed logarithmic scale and the centimetre scale mentioned in **1.431**. Cf. also the lefthand scale of figure 1, etc.

2 The engraver has done more abundant honour to the back parts (fig. 25) of the horse-power slide rule by reproducing it on a larger scale than the front (fig. 24), which also is shown somewhat enlarged; but this had the advantage of making the mean pressure scale more easily read in the figure than in the original instrument! (For the actual size, see **4.44**.)

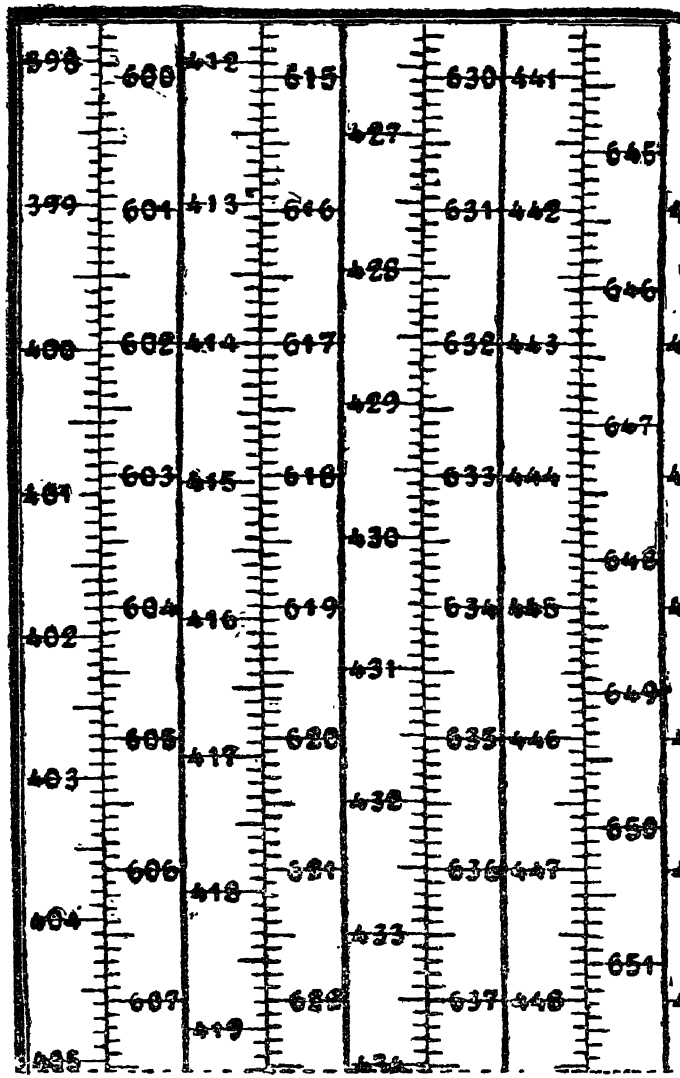


Fig. 36. Part of Tichy's Graphical Table of Logarithms.

(The length of page of the original table is evidently twice that shown above. In this space is given what occupies 28 lines of the same size in, say, Paterson's Five-figure Logarithmic Tables—including both logarithmic and anti-logarithmic tables; but the accuracy of the latter is not as great as that of Tichy's Table. And so the difference in actual space occupied by the two tables is not so much. In Chambers' seven-figure tables, 12 pages are required for the same range of the logarithmic table.)

When we consider how to represent this formula by a nomogram, we see that it is not one of any of the forms we have considered, and so we cannot say how the figure should be arranged. But it is easy to calculate from the formula suitable corresponding values of P, p and R. If then the scales of the two pressures are set out uniformly on parallel lines in opposite directions (fig. 37), it is easy by a series of intersecting lines to get the graduated line for R. Thus for the extreme value $R=1$, or 100°/₀, $P=p$ which gives one end of the R line as the intersection of all like graduations—a kind of centre of symmetry for the pressure scales. Again for $R=0{\cdot}1$ we get $p=\frac{1}{10}P\ (1+2{\cdot}3)$, whence come the pairs of values for P and p, 100, 33; 200, 66: the intersection of the lines joining these pairs of graduations gives the point $R=10$°/₀ on the R line. And so on, for as many graduations of R as we judge to be useful.

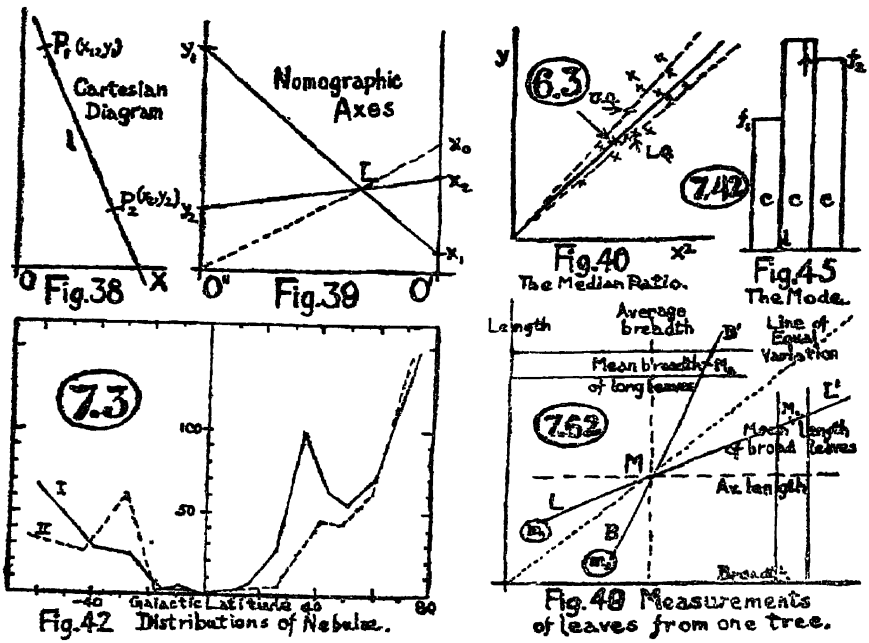
Mr. Kharegat, who with many details and alternative scales, such as engineers delight in, prepared me this nomogram, much reduced the amount of drawing required in constructing it by using the fact that the above formula belongs to a class for which the corresponding nomograms consist of three rectilinear scales two of which are parallel; this is the class (B) described in the Encyclopædia Britannica, 12th. edition, Vol. 31, p. 1142. Obviously, when $P=0$, then $p=0$, and therefore the R line is the join of the zero graduations of the pressure scales. Then, taking $P=100$, it is easy to calculate values of p corresponding to the values of R which are to the graduated: a fan of lines radiating from the graduation $P=100$ to the graduations of these calculated values of p intersects the line of zeros in the corresponding graduations. When these graduations have been marked, the construction lines are of course deleted.

Ex. 1. From the equation $c=h\frac{\cdot 0001635t}{1+\cdot 0001819t}$ construct the nomogram for the correction to be applied to the reading h of the brass scale of a barometer at temperature t (cf. **5.1** Note).

Ex. 2. The area A of a segment of a circle smaller than a semicircle is given approximately by the equation $A=\frac{1}{2}h^3/c+\frac{3}{8}ch$, where h is the height and c the chord of the segment. Construct a nomogram to give the result of this equation. (If the formula is written $A=h^2(\frac{1}{2}\,h/c+\frac{3}{8}c/h)$ it is reduced to the type here considered ——sometimes called **Z diagrams**: rectilinear scales for h, h/c and A compose the nomogram)

5.6. The Principle of Duality. Mention should be made here of another way of relating nomograms to Cartes-

ian diagrams which is sometimes illuminating.[1] In the latter figures we regard a line l as determined by two points P_1 and P_2; in the former this corresponds to a point L determined by two lines, p_1 and p_2. This can be easily worked out in figures 38 and 39; there the ordinates and abscissæ



[In figure 39 the lines (x_1, y_1), (x_2, y_2) should have been marked p_1, p_2 respectively.]

of figure 38 are set out as **parallel coordinates** in figure 39. It is clear that to every point on l corresponds a line through L: this is illustrated by the point of intersection with the x axis to which corresponds the line through the origin of the y scale in the nomogram.

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1 Cf. Journ. Biol. Chem, **59** 391. The principle has far-reaching applications in pure mathematics. A distinction may be made between the *point* (or *punctual*) *equations*, those we are accustomed to as determining which points are on the intersection lines in ordinary graphs, and the *tangential equations* of the corresponding points which lie on scales in alignment nomograms (Hezlet's "Nomography", p. 16)

Ex. Prove that the ratio of the distances of L from the O', O'' scales is equal to the ratio of the intercepts of l on the x, y axes; *i.e.*, all points which correspond to lines of equal slope are equidistant from either the O' or the O'' scale.[1] Obtain an expression for the distance of L from the line of origins $O'O''$.

(Note that OX corresponds to O'', OY to O', and O to $O'O''$.)

5.61. To apply this idea to **the body area nomograms** given in part of figure 1 and in figure 33, it is necessary first to modify the latter so as to transform the curves for equal area into straight lines. This can be done as in **9.4** by making the scales along the axes logarithmic; for, putting $X=\log W$, $Y=\log H$, the equation given in **5.3**, Ex. 6 becomes $\cdot 425X+\cdot 725Y=$ a constant, if the area A is taken constant as along any one of the curves; and this equation is linear in X and Y. Hence the curves of figure 33 become straight lines of constant slope, $-\frac{17}{25}$, on logarithmic ruling; and each of these lines corresponds to a point on the area scale which is parallel to the parallel reference logarithmic scales of the nomogram, and at distances from these scales in the ratio 17 : 25, if the units of the logarithmic scales are alike.

Ex. Convert diagrams like that described in **9.62** into nomograms with either uniform or logarithmic primary scales.

(Cf. Encyc. Brit. **31** 1141, figs. 7, 12; on this page also a more general account than that above is given of the principle of duality. Note that **determinants (1.83)** are made **the basis of classification** of nomograms in this article in the Encyclopædia.)

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1 For use later in connection with figure 64 and **9.521** (ii) we quote here from the Journ. Biol. Chem. **59** 397: "On the nomogram broken lines define the position of the points of intersection of arterial and venous lines corresponding to values of the respiratory quotient in the range between 0·70 and 1·00. Each of these lines is the locus of all points correlative with lines of a definite constant slope on a Cartesian nomogram. But on a Cartesian nomogram, having total oxygen and total carbonic acid (**9.2** f. n., etc.) as coordinates, the slope of a line joining arterial and venous points measures the value of the respiratory quotient (**9.1** Ex. 3)."

CHAPTER VI

TYPICAL NUMBERS

" The true Logic for this world is the Calculus of Probabilities, the only Mathematics for Practical Men."

(JAMES CLERK-MAXWELL.)

6.1. COMPARISONS: There are several ways in which we think of the relations between things—for individual things we think of their differences or their ratios. When we wish to compare groups of things, such as the coins in the purses of several persons, addition of the values suggests itself as the most obvious method. But this method we should consider rather out of place if we had to compare several patrols of boy scouts. For one thing, the possibilty that the patrols contained different numbers of boys suggests that a fairer method of comparison than, say, addition of heights would be to find the arithmetic mean of their heights.

But when we look at the boys in their patrols sized from right to left, we may feel that simply to give the average height is not as fully true a comparison as we should make. Some patrols appear very even, others slope down very steeply to their left, while occasionally the line of heights is quite broken at either end, by a giant on the right, or, it may be, by a very short, but very sharp, youngster on the left. If we feel we must reduce such facts to figures (and there are similar, though less " human ", facts that are well reduced to such a manageable form), there are two things we may do, besides finding the average.

6.11. Having arranged the boys in **order** of height, we may decide to take the middle height (or the mean of the two middle heights if the number of boys is even) as typical of the whole. The reasonableness of this may be seen more clearly if we consider another comparison,

say, one between the wealth of people who dwell in different parts of a city: comparison by taking the arithmetic mean of the wealth of a number of people selected at random from each district may be quite fallacious if, say, in a poor district there has been included the wealth of a millionaire who is content to live in a humble way. If, instead, the numbers representing wealthiness are arranged in order of magnitude and the middle one taken, the abnormal influence of exceptional extreme numbers is avoided, and a more truthful typical number than the arithmetic mean in this case is obtained. This is called the **median.**

If you test a few series of actual measurements of any kind (*e.g.* those in Table I) and compare the median value with the arithmetic mean, you will find that in most cases they differ but little; and hence you will feel that the median is quite adequate to give us what we have expected from the mean value, and you will have further the satisfaction derived from the comparative ease with which in many cases it is obtained. But even more important is it to notice that here we have introduced quite a new principle in that we are using *order* of magnitude to guide us to a typical magnitude, and not magnitude itself.

The idea of *order* has become prominent in higher mathematics, and it is interesting to find a practical use for it in elementary work.

A. A. Robb has constructed a theory of space and time based on the ideas of "before' and "after": times, points, lines, and planes are known only in their relations of succession, not fundamentally in their identity or equality; but the "order" used is called "conical" order, to distinguish it from linear order.

Ex. What is the median value of $\frac{\tan\theta}{y}$ in the example of **2.21**? Does it differ much from the mean value?

6.12. But this does not enable us to differentiate between the patrol which is very uniform as regards height and that in which there are great differences. To do this we must seek the aid of quantity again, and find the **deviation** of the heights from either the mean or the median value. The idea of this is almost obvious. The difference between each measurement and the mean is found, and the sum of all these (obviously without respect to sign) is divided by the number of measurements to get the **mean**

deviation *from the arithmetic mean*: so also for the mean deviation *from the median*. This gives an acceptable measure of the differences within a group, while the A.M. or the median indicates the position of each group with respect to others; and so from the two numbers, a typical height, and a deviation from that height, we get a fairly vivid idea of the appearance of each scout patrol.

6.13. We have used scout patrols as a simple illustration. But if one is to think fairly of statistical work, it cannot be too carefully remembered that the word "**typical**" becomes full of meaning only when the number of things of which it is a type is large. If the type of a large number of things is well chosen, there is a large degree of certainty that a particular individual thing will resemble the type closely[1]: but, when the numbers are as small as those in a scout patrol, little can be said with the confidence of certainty behind it; and this would still be true with less uncertain things than boys! In statistics the "**universe**" of things considered should be **as large as possible**: how good, manageable samples should be taken from that universe is a difficult question which is examined in text-books on statistics.

6.141. Another typical number is the **mode**, which is simply the value that occurs most often in a large group of measurements—it is the position of the maximum in the frequency curve (Chap. VII), the "fashionable" measurement. But we do no more than notice it here, though it may be very reasonably regarded as the proper typical number from which to measure deviations. Relations can be found between the positions of typical numbers: cf. **7.42.** The *geometric mean* (**2.21** Ex. 2) is rarely, and the *harmonic mean* scarcely ever used.

6.142. As to measures of deviation, there are others than the mean deviation which are employed because of convenience in calculation or for other reasons: these are much used in investigations, but they lead us into more difficult regions of thought and manipulation than we may enter here. We consider only one of these measures, the **quartile deviation**, in **6.3** and there it will be seen that this measure of deviation is rather arbitrary, though quite in accordance with commonsense standards. The **standard deviation** is mentioned in **7.52.**

6.15. The consideration of measurements made in this complex world by the imperfect means with which we are equipped must raise many difficulties:

1 This of course does not exclude the "practical certainty" that in a large number of measures there will be *some* that are very different from the typical number. See, *e.g.*, the rainfall figures in **6.52** Ex. 2.

through these difficulties scientists are just feeling their way, and many non-mathematical scientists simply follow empirically the methods which have been devised by statisticians to give meaning to measurements. There is often no proof of the "correctness" of these methods save in **the intelligibility of the results** they give. But it all makes the study of this branch of mathematics more of an adventure; the thrill of seeing for oneself the vistas that great intellects have descried (or merely indicated, for intuition[1] is the way of the great in Mathematics as in other fields) is exciting enough; but it is possible for any of us, who are not content with merely travelling through the carefully mapped regions of the text-books, however interesting, to experience the thrill of finding sure ground through the many difficulties and uncertainties of the application of quantitative ideas to everyday occurrences: we have only to be on the look-out. A key to the meaning of the gusts before the rain-cloud bursts, or to the way in which advantages are balanced when prices are fixed, may be given us at any time!

6.2. TYPICAL RATIOS: These ideas may be applied not only to single measurements but also to pairs of related measurements; for the ratio of these, just as any other function, may be regarded as an indirect measurement. But this statement has a special meaning when we use mechanical aids in dealing with the ratios of these pairs of measurements.

6.21. Already with the help of the simple **slide rule** (**4.13** Ex. 4) we have worked out something like the converse of this: given a number *e.g.* 1·414, which we may regard as typical, we have found pairs of integers whose ratio is approximately equal to this. And so, if we are given a number of pairs of measurements whose ratio we know should be constant, *e.g.*, measurements of the circumferences and the diameters of coins, we can find opposite the end of the slide *a good value* for that ratio by moving the slide about till we judge the pairs of measurements to be on the whole as nearly opposite one another as possible. This of course leaves a great deal to the judgment of the user of the slide rule, but it is a device well worth practising just because of the speed with which it operates, and its freedom from arithmetical errors (cf. also **1.33**).

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1 Mathematicians like S. Ramanujan, have often stated propositions, of the truth of which they were confident, though they could not prove them.

This method can be extended to more complex relations between two measurements if a suitable slide rule is available; *e. g.*, to pairs of numbers with a constant product xy (as for Boyle's law) if the slide is inverted, **4.16**, to pairs for which x^2/y is constant with the ordinary four-scale slide rule, and to pairs for which x^2y is constant if the slide is inverted in the ordinary slide rule; and so on. (See any book on the slide rule, or Horst von Sanden's "Practical Mathematics" pp. 23, 24. Cf. also **9.45** Ex. 2, etc.)

In using this method we are really considering the *order* of the graduations with which we are concerned, arranging them, some on this side, some on that side, in a balanced way; and so the result we get is of the same nature as the median value of the factor that expresses their "constant" relationship.

6.22. A less empirical, though more cumbersome, way of arriving at this median value itself, without performing the arithmetical calculations, is to **plot the** pairs of **measurements** in a way that suits the special problem before us. Suppose we are given pairs of values of x and y which we know lie nearly on a curve $x^2=ky$. Each pair of values of x and y substituted in this equation gives us a value of k, and from among these we have to choose.

Instead of working out all this arithmetically, arranging the values of k in order and choosing the middle one, we set out the problem in a much more striking way by carefully plotting points (x^2,y). We thus get points more or less on a straight line through the origin; if each of these points is joined to the origin we get a narrow fan of lines, the slope of each line being a particular value of $1/k$. But it is better not to draw the lines. Instead, we pass a fine thread through the origin and each of these points in succession, beginning with the least slope; we really are arranging the values of $1/k$ in order of magnitude. We know the total number of points, and so it is easy, as the thread passes over the plotted points in succession, to stop at the middle point. The calculation of k from x^2/y for this point gives us the required median value. (Fig. 40.)

In some respects the method of **1.33** is more general, for the trend of the points plotted need not be through the origin: but this absence of a fixed point makes it a very arbitrary proceeding to determine a line which is in a median

position with regard to all the points. This, however, is what is really done in **1.33**; and there is no reason why a skilled experimenter should not estimate, in accordance with the idea described in the next paragraph, the positions of quartile lines through a linear assemblage of points, if that would serve any useful purpose.

6.3. QUARTILES. In connection with the graphical method of the last paragraph it is quite convenient to introduce the second measure of deviation referred to in **6.141**, though it has no special connection with fans of lines. It is obtained by considering the "scatter" of the values below and above the median value. The method is just a repetition of that for finding the median: the middle value, in order, of the measurements below the median is called the *lower quartile;* that for the measurements above, the *upper quartile.* This gives us at once a good idea of how the values are distributed, whether close together or far apart; and the fact as to this distribution can be stated adequately and concisely by noting the difference between the quartiles, the inter-quartile range. A measure of deviation commonly used is the **semi-interquartile range,** or **quartile deviation;** it will be seen in most cases not to differ much from the mean deviation from the median (and that from the mean also): it is also called the **probable error**[1] of a symmetrical series (Bowley's "Statistics"[4], p. 113); but this term is given many meanings: cf. **6.4.**

This deviation is shown graphically in figure 40, where for clearness a greater "scatter" among the 15 measurements is shown than ought to occur in a good series. The point which gives the median value is shown by a long arrow, the quartiles by short arrows marked *L.Q.* and *U.Q.* respectively.

NOTE.—In this diagram there is also indicated what often happens in carrying out experiments: large values of the constant sought seem to depend on the size of the absolute measurements made; here $1/k$ is represented as large when y is large. When this occurs, it is necessary to examine the way in which the experiment is being conducted; if nothing faulty can be found in the procedure, it may then

1 On the same page, 310, of Yule's "Theory of Statistics" as is referred to in **7.55** is explained how the probable error is found to be 0·67448975 times the standard deviation (**7.52**) (or standard error). "Probable" signifies that it is equally likely that an error which occurs will be greater or smaller than the probable error.

be concluded that there are limits to the absolute size of the measurements which should be made in trying to find the value of the constant.[1] Thus in finding the value of π by direct measurement very small circles and very large circles are not used because both are so difficult to measure with accuracy. Perry puts it thus: "the percentage error or the probable error may be in some curious relation to the observations."

6.31.[2] There is no reason, save convenience, why in considering values at a given distance along an ordered series we should restrict ourselves to the fractions $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$. It is sometimes convenient to note the values at $\frac{1}{10}$, $\frac{1}{5}$, $\frac{3}{10}$,..... along the series, and these values are called the first, second, third,.....**deciles.**[3] So also we might speak of **centiles,** the values at intervals of $\frac{1}{100}$. These typical numbers are sometimes useful in making general comparisons: thus "in the statistics of *wages* the upper[4] decile is always somewhat less than twice the median.....In the distribution of *salaries* the upper decile is approximately twice the median.....the prevailing distribution of *income from property.* In the Massachusetts probate statistics the upper decile is eight or nine times the median..... Among French estates the upper decile is thirteen times the median.". Cf. also Bowley, *op. cit.* pp. 68, 70.

6.4. Hitherto in this chapter we have been concerned mainly with the relation among themselves of numbers in a series. But we might have proceeded directly in the problem we first stated by **comparing sets of values two by two,** found some relation between each pair, and combined these relations; instead of, as we did, trying to find

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1 Chemists sometimes speak of *optimum values* of the substances to be used in an experiment; these are related to the size of the weighing machine which has been found most sensitive and consistent: and this depends on the dimensions of our hands and eyes: and so after all it is only commonsense!

2 The remainder of this chapter is not required in Chapters VII and VIII.

3 Galton plotted at equal intervals the values of a series in the order of their increase: from the curve through these points, the above typical numbers can of course be read off at once (Bowley, *loc. cit.*, p. 107; **7.31** f. n.) This curve is called the **ogive** because of the sudden turns at either end. Cf. **9.441.**

4 Presumably the ninth? Or is the reference to quartiles? The quotation is given in Pigou's "Economics of Welfare", p. 698. Cf. **9.42.**

quantities which were characteristic of each group. The former is often a very natural procedure, especially when we have some standard series with which we can compare others. Thus in **8.4** we find the departure of the heights from the correct heights we can calculate, and the **average error** we thus get is used as a measure of the correctness of the performance. Similarly, Professor Perry in finding the formula which fits observed values very closely (**1.33**) writes of "the probable errors in the observed values", and takes for granted that it will be understood that "probable error" (*not* the P. E. of **6.411**) means the difference between what the value in question is and what it ought to be according to the probable "law". (Cf. also **9.33** Ex. 6 *Note.*)

Differences alone are sometimes what is of importance in a set of values: thus what is recorded in meteorological reports is how far rainfall in a stated period has departed from the average for that period. In the Census of India, 1921, VIII 119 are given charts which show the inequality in the numbers of men and women at all ages by the departures of these numbers from 50% of their total. The superiority of this type of diagram over the more usual arrangement, in which the total numbers of either sex are represented (e.g., Whipple, "Vital Statistics" p. 194), is evident : cf. **7.11**. Rainfall figures are dealt with thus, *loc. cit.* VIII 12. Cf. also **2.311**.

Ex. Find the average error of the formula you have fitted to one of the curves in figure 8 (**2.0** Ex. 2).

This method can be applied to marks of students in two examinations marked on the same scale; but in such a case (just as in comparing by pairs the heights of groups of boys), **neither set** of values is **a standard** with which to compare the other, and the result is difficult to interpret : *e.g.*, if we seek a distinguishing meaning in the average difference between pairs (account being taken of sign), we find it is just the difference of the A.Ms., which is a rather jejune criterion ; at the most it gives a comparison of the standards of examination. The matter is essentially very difficult : consider the ways in which inconsistency between results may arise—in the candidate, a spurt of over-confidence, illness or health, his temperament, the nature of the questions; in the examiner, inexperience or weariness, obscurity in answers, *his* temperament (the more stable, it is hoped!); and then, errors in handling the figures.

This is very depressing : it suggests to those who groan because of the unequal lots that are man's in this world, that man himself would not have improved matters had he had the ordering of the universe ! Certainly it is true that individual scores may be very far from representing the truth ; and consequently resort should be made on every possible opportunity to sources of information other than mere examination totals—we readily acknowledge that they are far from representing life-values truly.

6.41. But we need not leave the matter there. Methods have been developed of handling facts of this kind if they are sufficiently numerous; the various factors we have suggested above as causing discrepancies tend to compensate one another in ways that can be defined. *On certain assumptions*[1] **many formulæ**, more or less empirical have been worked out. These are frequently applied in the non-mathematical sciences, sometimes without much attention to the underlying assumptions, which are mathematical simplifications of the data that are often difficult to relate to the actual situation.[2] Here all that is attempted is to give an idea of the type of formula that is available for elucidating the nature of numerical data.

If we arranged the boy scouts in two equal patrols in order of increasing height, and found the difference between the heights of each pair of boys, we would get a fairly clear idea as to whether the distribution of height in each patrol was alike, whether heights increased correspondingly on the whole or not. This is a procedure not unlike that in the **method of ranks**,[3] which may be used in finding the degree of consistency between two examination tests; only in the latter *no account is taken of the actual size* of the *n* results of

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1 *e.g.*, regarding the form of the frequency-distribution : cf. **7.31**, **7.32**, **7.42** (p. 120).

2 Cf. Brown and Thomson, "Mental Measurement", p. 103.

3 This method may be looked upon as midway between what Yule distinguishes in his "Theory of Statistics" as the theory of *attributes* (qualities, types, we may call them : cf. **7.211** f. n.) and the theory of *variables* (in which there is a definite order for the groups) : cf. Brown and Thomson, *op. cit.*, p. 131.

each of the tests arranged in order of magnitude. The calculation in this method is made in two stages: first is found the number ρ, the *rank correlation coefficient*, from the formula $\rho = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$, where d is the difference of the ranks for each individual in the two tests; and then is found from $r = 2\sin(\rho\, 30^\circ)$, the *correlation coefficient*[1] r, which is a measure of the degree in which the two series agree, $+1$ if there be perfect consistency, 0 if they be indifferent to one another, -1 if they be perfectly contrary to one another (cf. **7.622**).

Ex. In Table I are given three sets of marks obtained by students at a test examination and at a public examination not long after: *f* signifies a failure at the latter examination, the marks earned not having been announced. Verify that by using the above formulæ (neglecting fractions of ranks) the values of r for A and B are ·700 and ·788 respectively. (This indicates a rather higher consistency than a mere inspection of the figures suggests, though a still higher consistency should be possible.) Test the marks in some of the ways indicated earlier in this chapter e.g., by reducing the marks to percentages, finding medians, etc.: space is left for doing this. The marks in C are given for further practice.

(The results of A seem more satisfactory from the point of view of those who, conducted the test examination in that only three of the candidates failed at the public examination; but, if the public examination is taken as the standard, the coefficients of correlation indicate that examination B was a more reliable investigation than A of the ability of the students. But this comparison of the values of r is unjust: for in ranking the larger number of failures in B_2, they were arranged in the same order as they had at the test examination. In the absence of the marks scored at the public examination by the candidates who failed, the only way to make a true comparison is to include in the calculation only the marks of successful students.

Another possibility of vitiating the comparison is the fact that the last six candidates in A, and the last eight in B, had failed at a former examination and were not regular students. The marks of these candidates noticeably fluctuate, whatever the reason, and perhaps they should be omitted from the comparison.

If the numbers given are numerous, they should be written on **cards**, one for each individual, and these can be **sorted** in any desired order. But in a case like that of Table I the following **method of noting the ranks** is effective. Working from the highest score downwards, choose a convenient mark, and put a dot, or other small sign, opposite the marks (three to eight, say), which are higher than that chosen. The rank numbers can then be easily assigned to these few

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1 Other approximate values for r are given in a convenient way in Yule's "Theory of Statistics", p. 202 f.

TABLE I—Consistency of Examination Marks

	A_1	A_2	B_1	B_2	C_1	C_2
Total	*825*	*625*	*725*	*625*	*700*	*675*
	192	277	250	299	329	337
8	278	334	179	236	332	379
15	249	338	217	*f*	262	344
.	227	245	198	*f*	322	337
19	244	317	193	*f*	297	352
	173	223	166	*f*	305	355
	188	220	170	*f*	272	323
	202	220	286	350	380	324
	198	257	449	430	355	378
5	330	343	178	266	280	343
	199	292	267	279	263	293
21	232	269	222	253	299	315
1	436	427	437	375	289	333
	218	275	367	388	240	*f*
	213	*f*	229	274	269	*f*
24	229	321	297	308	249	*f*
	175	211	339	331	322	*f*
	205	*f*	258	280	292	306
12	254	300	176	266	257	317
3	347	364	292	283	262	292
20	242	282	280	326	306	293
	199	251	239	*f*	269	312
	218	330	326	296	423	406
	192	245	228	265	259	271
10	274	279	229	271	274	337
17	247	275	251	387	376	407
	174	261	301	347	327	322
	209	267	222	*f*	361	386
	199	277	308	320	263	368
4	345	377	226	232	334	368
15	249	300	248	262	293	311
.	227	263	290	320	287	300
	197	239	197	*f*	299	368
11	267	280	272	*f*	319	335
13	251	306	296	359	369	378
21	232	241	202	258	243	297
	219	351	188	241	360	389
6	295	353	230	*f*	342	380
.	220	237	396	376	270	*f*
	180	249	216	*f*	258	*f*
.	225	290	205	*f*	228	*f*
2	350	302	428	422	293	*f*
	196	299	385	393	252	*f*
.	229	279	163	*f*	237	*f*
.	229	266	262	299	390	364

values and written so as to obliterate the dots. If the process is repeated down the series, the ranking is effected with a minimum of strain on the attention. The method is indicated in column A_1. Note that if identical values occur an odd number of times, the middle rank is repeated that number of times.)

Ex. 2 Some of the marks in Table I give the impression that at the test examination higher marks were given on the whole to the better boys than they earned in the public examination—possibly due to unconscious bias in the teachers who conducted the test. Does an examination of the marks, by the methods of Chapter VII or otherwise, confirm this impression? Is there any way of saying that one examination is more reliable than the other?

6.411. It is worth while examining the vague statement in **6.41** about the procedures in comparing heights and marks. The object of the comparison of the patrols by means of the heights of the boys is not to find which patrol is the taller on the average, but to test how far height is a criterion by which to establish the similarity of the patrols. We have no right to expect that height will be a good criterion, though it may be such. In examination tests we have a right to look for consistency in the marks of individuals on the whole. It is just because of this distinction that in one case we compare the measurements of *pairs of individuals* who have to be arranged to give some sort of regularity, and in the other we consider the serial orders of the marks of **the same individual.** The questions we ask are: does the arrangement by height give consistency in change of height? and, do two tests of abilities, which differ in a

(contd.)	A_1	A_2	B_1	B_2	C_1	C_2
9	277	320	171	236	302	339
15	249	291	179	f	240	291
	217	243	240	f	245	286
18	246	305			250	303
7	294	361			216	316
	177				398	429
Additional candidates						
			227	258		
			276	342		
	185	276	176	f		
	174	275	202	278		
	192	268	165	281		
	186	251	186	275		
	155	239	190			
	182	293	172			

definite way along their range, give consistent results for the whole range of abilities of the candidates?

Remember that in the case of the marks we are investigating merely the consistency of two sets of measurements; what the characteristics of each set are we do not ask. This would be our question if, say, we considered *how often* the difference of the two marks (adjusted to a common standard) were positive or negative—we add the differences in some way: if we considered *how great* these differences were, we would get merely a more precise answer.

Investigation of the character of a group as a whole is the method followed in **7.31** Ex. 3, where a difficulty arises as to whether ability or the test should be taken as the standard. There irregular variations in ability are noted, but deliberately set aside. The question is as to whether a large area in a graph means unusual ability of candidates, or unusual ease of tests: the difficulty is as to aggregates. Here in **6.41**, it may be repeated, the difficulty is as to consistency of measurements—whether in one case height, the type of measure chosen, is such as to show consistency, and how far in the second case the marks show the consistency they should show.

The possibility of treating statistics in a variety of ways is a matter you should ponder repeatedly in special cases. In the simple case discussed above it may quite fairly be held that the comparison is a bad one; for the procedures are the converse of one another in respect of the origin of the pairs of measures compared, the nature of the measures, and also in that in the second case one series is taken as a standard of rank; cf. also **6.52**, where differences *along* a time series are considered. It may be held too that the way suggested for comparing the patrols is essentially valueless. It is certainly well worth your while to form a considered opinion on the matter, after you have tried to get results from the actual figures.

This objection to a standard would have apparently more cogency if the formula we had used were Spearman's "footrule", which gives in an easy way from their ranks a rough idea of the amount of consistency between two sets of numbers. **Spearman's coefficient** is expressed by $R = 1 - \frac{6\Sigma g}{n^2 - 1}$, where Σg is the sum of the "gains" in rank (sum of positive differences) of the second series on the first. A discussion

of this formula will be found in Nunn's Algebra II 472, 488. It may be tested on the marks in Table I, for *r* = A, Sc. (R 459).

Ex. Consider how far *quantity* and *order* enter into each of the procedures discussed in **6.411**.

6.42. But it is not enough to calculate correlation coefficients. The value found may or may not be significant. A test of this is given in the **probable error** of r which in this case is defined by P.E. $= \cdot 6745\,(1-r^2)/\sqrt{n}$.[1] In the above example this works out to ·045 and ·036 for *A* and *B* respectively, and the comparative smallness of these values indicates that the correlation is a reality: a criterion given for the reality of a correlation of any degree is that r should be at least six times its probable error.

6.51. Thus far in this chapter we have considered only sets of numbers occurring simultaneously. When we consider series of values which are measured concurrently in time, a new consideration emerges much more clearly—that of **cause and effect.** We have tried to define the nature and extent of similarity or contrast between sets of values. Can we now determine how far similarity or contrast in the variation of two series of numbers indicates some causal relation?

Sometimes the relation is obvious: the following represent, at fortnightly intervals during the year 1914, the percentage f of plague-infected rats examined in Bombay, and d the plague mortalities for the same intervals:

f	·8	1·2	2·0	2·7	3·2	3·5	3·1	2·5	1·7	1·2	0·7	0·4	0·4	0·4	0·2	0·3	0·3	0·3	0·2	0·2	0·2	0·1	0·1	0·2
d	2	3	9	18	42	52	56	32	25	13	6	1	3	2	1	5	0	0	0	0	0	1	0	2

If curves are drawn representing these figures, the fact that the curve for plague deaths repeats the curve for rat fleas after a certain interval is very apparent: this is so especially if the vertical scales are adjusted so that, say, the maximum

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1 The proof of this formula is much beyond our scope: cf. Nunn, *op. cit.* II 500; the fraction is that mentioned in **6.3** f.n. Nunn finds a probable error of $\frac{0\cdot 43}{\sqrt{n}}\left(1+\frac{1}{2n}\right)$ for Spearman's coefficient.

on each is the same—the **lag,** the interval in time between corresponding points on the curves, could then be measured as the *average of the differences of the abscissae* for equal ordinates at regular intervals along the curves. Or, the time-origin for the mortality curve may be *set back* (**1.21**) by an amount equal to the lag and then the closeness of coincidence of the two curves may be observed directly: this is a satisfactory device, for there is little meaning in measuring the lag with great exactness—the relation of cause and effect is not usually simple and unique. A good, though laborious, way of determining the lag in more obscure cases is mentioned in Ex. 2 of **6.52.**

6.511. These numbers may also be compared as such without a graph if they are reduced to a convenient common scale, say, percentages of the mode or some other typical number in each series. The resulting numbers may be regarded as **index numbers** of each variable. In this form, after the correction for lag has been applied, the numbers may be compared by their average error; and this may be taken, as in **6.4**, as an inverse measure of the closeness of their *correspondence,* or *fit,* or *correlation,* whichever word you chose to use.

6.52. If we seek a standard number by which this correlation may be expressed, there are several such which are not difficult to calculate; but the meaning of these is very difficult to state with exactness.[1] In books on statistics information is given which will determine how far a characteristic number is suited to a particular type of problem. One such number is the **coefficient of concurrent deviations;** this is $r = \pm\sqrt{\pm\frac{2c-n}{n}}$, where c is the actual number of concurrent deviations out of a possible number n; in some cases it gives a rough indication of the correspondence between series of numerical observations. It is described as

1 In the Census of India, 1921, VIII cxvi is given a graphical description of a case of inverse correlation where the result of calculating the coefficient of correlation was judged to be misleading!

"a coefficient of correlation which has the merit of extreme simplicity, and in most cases may be used satisfactorily in the study of short time oscillations[1] (cf. **2.311**).....it takes almost no account of the general trend.....well suited for use with irregular graphs.....in which smoothing by means of a moving average is well-nigh impossible."[2]

The *deviations* here considered are the increases or the decreases of successive numbers in both series: if two increases or decreases occur simultaneously, this is called *a concurrent deviation* of the measurements. The main idea of the formula is made obvious by considering the extreme cases, when there is nothing but coincident changes and when there are no such changes. Expressing this algebraically, the three pairs of value of c and r we have to fit into the formula are respectively $n, 1$; $\frac{1}{2}n, 0$; $0, -1$: the middle values determine the form of the numerator. There is no such clear reason for taking the square root, but this is frequently done in defining typical numbers. In this formula the upper or lower signs are taken together, for a reason that is easy to see.

Ex. 1. When this formula is applied to the flea, plague figures given in **6.51**, a negative value of r is found for the figures for the whole year, whatever the lag be. (This is obviously due to the inclusion of figures for the second half of the year when no definite cause was operative, and only random cases of plague, probably imported, are recorded.) The figures for the first six months alone give practically perfect correlation, either with or without lag. Test these statements, and state what the indifference to lag in this case signifies for the formula.

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1 This remark is not quite supported by Exs. 1, 2 below; but note the words "in most cases".

2 King, "Statistical Method", pp. 207, 211 : but no specific reference to the original authority is given which would enable us to examine applications of the formula. The insufficiency of considering only the sense, and not the magnitude, of the deviations is brought out well by the following clear statement of principle, given in the article from which Ex. 2 is taken : "If every fluctuation in the rainfall, appropriately measured were followed by an exactly equal, appropriately measured, fluctuation in the incidence of cholera, then the coefficient of correlation would be unity. If there were no relation between increases in rainfall and increases in cases of cholera, the correlation coefficient would be zero. If increases in rainfall were followed by equal decreases in cases the coefficient would be -1."

Ex, 2. The following numbers give x the rainfall in the catchment area near Poona, and y the deaths from cholera in Poona City from June 7 to August 31, 1914 : figures up to October 31 are added for comparison.

x, Daily Rainfall (in cents, ·01″) from June 7.

0	12	18	0	0	37	0	43	165	40
28	0	0	22	173	120	115	40	12	71
247	276	93	81						
July				49	93	.61	61	74	239
220	462	368	487	267	201	187	120	201	224
647	659	508	711	829	383	293	589	227	325
219	241	310	239	218					
August					343	284	206	420	591
422	398	187	87	51	59	87	77	31	90
12	15	12	31	82	94	145	85	151	146
42	12	25	37	34	7				
September						0	0	0	0
14	0	1	0	42	41	22	52	66	28
43	276	311	291	55	18	0	0	0	4
0	30	44	15	0	19				
October						0	13	105	76
1	0	0	0	26	0	19	0	0	7
5	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0			

y, Cholera Deaths from June 7

1	0	0	0	1	0	0	1	0	0
0	0	0	1	1	0	2	1	1	1
0	2	0	2						
July				2	0	0	0	5	7
1	1	1	5	6	0	4	4	2	6
6	7	8	6	6	6	5	10	6	10
12	11	14	8	6					
August					8	9	4	2	7
2	4	1	2	3	10	6	2	3	1
4	2	2	3	0	0	1	0	1	1
0	2	0	0	0	0				
September						2	1	0	1
1	0	1	0	0	1	0	1	0	0
0	1	0	1	0	0	0	2	0	0
0	0	0	0	0	0				
October						0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	2	0	0	0	0	0
0	1	0	0	1	0	0			

By using the standard formula for correlation (not discussed in this book) the following coefficients of correlation r were found between the rainfall figures (taken however, right up to Oct. 31) and the number of deaths 0, 1, 2, 3,.....13, 14 days later :

·58 ·63 ·58 ·57 ·64 ·63 ·67 ·73 ·75 ·78 ·76 ·66 ·62 ·62 ·51

(The value of r for an interval of nine days being the highest, this interval was taken as the lag; and the reality of the correlation was tested by an investigation as to the possibility of infective germs reaching Poona in that time. The corresponding P. E.s **(6.42)** were

·04 ·03 ·04 ·04 ·03 ·03 ·03 ·03 ·02 ·02 ·02 ·03 ·03 ·03 ·04.)

Find for the figures from June 7 to August 31 the corresponding fifteen coefficients of concurrent deviations. If you find that these coefficients do not confirm the above result, consider any possible defects in the method of concurrent deviations or in your application of it: *e.g.*, whether it is right to make "no deviation" uncommon in the rainfall series by measuring up to ·01"; whether the figures should be smoothed first by a moving average of, say, about a week; cf. **9.33** Ex. 3; how "no change" should be reckoned—taken as one of three kinds of coincidence, or taken as a coincidence with both increase and decrease, or excluded from n altogether. Make a thorough search for there *is* a connection; cf. **9.4** Ex. 5: Laplace defined *Probabilities* as "good sense reduced to calculation."

(Note that the number of pairs decreases by one for each day of increase of the lag reckoned. The comparison may be made very easily by writing in order on two pieces of ruled paper the signs for increase, decrease or no change, +, −, or ., regularly spaced (three or four between successive rulings), and by placing the paper showing the deviations of cholera mortality over that showing successive deviations of rainfall, and marking in columns on the former the coincidences for lags of 0, 1, 2, 3,....., 14 days. The mortality paper is moved one place down for each increase of the lag allowed for. In this way a record is got of the places in the series where coincidences are most frequent in each case. The comparison may also be made fairly easily from a table of signs arranged regularly in tens, like the x, y tables.

For the reason given in Ex. 1 the low figures of rainfall and plague mortality from September 1 to October 31 are not included in the above. But they are added in the x, y tables, so that it may be judged from the figures what is the effect of their exclusion: for reasons connected with the investigation into the disease it may have been necessary to include them.)

CHAPTER VII

FREQUENCY GRAPHS

7.1. CHECKING IMPRESSIONS: We must now consider how the chief ideas suggested in the last chapter can be applied to large numbers of measurements—so numerous that their addition, or even their arrangement in serial order, would be very laborious or impossible; *e.g.*, we cannot find directly the typical height of *all* people of a certain age; we have to take a very large "sample" of such people, and even then the mass of data we accumulate by measurements may be difficult to deal with simply by use of the ideas we have examined so far.

In chapters II to V we have generally considered how to deal with quantities which change in some regular way (in accordance with a "law", it may be said), which is known. This regularity has been expressed by some formula of greater or less complexity, and we have discussed methods applicable to all such formulæ. But there are many relations we meet with constantly which change in no obviously regular way, but yet are recognised by everyone as controlled by definite forces. Take the heights of children 10 years of age; they are not alike, and yet we have a general idea as to what height a child of that age "should be", as we say; *i.e.*, we acknowledge that there are forces and conditions which determine in a fairly definite degree what the height at a given age is likely to be. We can investigate the matter further, *e.g.*, by saying that parents of average size in a certain community will have children of equal heights at that age; but we are not much surprised if we find our prediction falsified, and we are very ready to acknowledge that the forces which determine such ordinary things are too numerous, or too elusive, for complete description.

Are we then to give up thinking of such things? Are our general impressions, which we all share, mere illusions? We have illusions; and so it behoves us to apply the test of

measurement and **arrangement** to our impressions, or rather, in the first place, to what they are based on. Lord Kelvin has said: "When you can measure what you are speaking about and express it in numbers, you know something about it, but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind." You perhaps rise at once to contradict this, or at least to say that it is only a partial truth. But it is a saying that goes deep enough to treasure with you and ponder over at least for your student-days.[1] We may hope to see more deeply into it now.

7.11. There is a graphical method of representation of numerical facts which we need not consider here. By means of areas, it may be, of various shapes, sizes and colours a striking idea is given of, say, the exports and imports of a country, and how they should be analysed and compared with those for other countries. This is mere graphic representation of facts, **drawing** certain quantities **to scale**; and this representation cannot be used to deduce anything further as to the nature of these facts. For the purpose it can serve, however, it has been very effectively employed in educational films, where there is the added advantage that by change in size and shape and colour the development of fact can be presented graphically and "kinematically". The charts in the Bombay Government's Report on Wages and Hours of Labour in the Cotton Mill Industry are mainly of this nature: the material in this Report which could have been presented as frequency curves is given in tabular form, probably because people trust figures more than curves. (Cf. **7.4 Ex. 1.**)

Many most ingenious devices have been used to represent facts graphically. A most striking example is the "Butterfly Diagram" which shows the variation in the numbers and positions of sun-spots from year to year. This diagram is frequently reproduced in books on astronomy, and may be seen in Hutchinson's "Splendour of the Heavens", p. 146. (This work is a veritable mine of ingenious and interesting diagrams.) Sun-spots have been studied very thoroughly by the use of the methods suggested below; and they should be noted also as affording a good illustration of *periodicity*, a subject, however, that is not relevant to this chapter: cf. **2.311.** See *e.g.*, Sampson's "The Sun", Ch. VI (Cambridge Manuals): also compare **2.33.**

A good contrast between methods of representing the numerical relations between the sexes is given by the diagram in Whipple's "Vital Statistics", p. 194, and the more striking, though redundant, graph in the Census of India, 1921, VIII 119: these are really frequency curves; cf. p. 114 Ex. 5.

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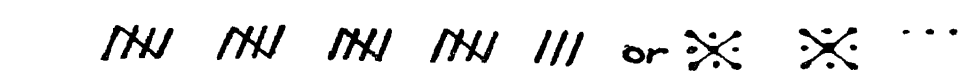
1 If you have not been subdued by the comprehensiveness of the motto at the beginning of Chapter VI, your revolt against it may have been still more severe. But it is a more penetrating statement even than this, and it should be pondered with care——even to the capital letters!

7.2. QUANTITATIVE CLASSIFICATION: The first step in dealing with a large number of quantities of the same kind is to classify them. In a sense this is done when we arrange quantities in a time series or in any other obvious way, *e.g.*, the numbers of people who visit an exhibition on successive days. When there is no clear way in which to arrange the given quantities, we have to create classes in any way that we think convenient and find the **frequency** with which the given numbers occur in each of these classes. In this method time is seldom the variable with respect to which we classify: usually we consider graduated measures of objects existing together at one time.

The way in which we chose these classes of course may make a great difference to the result we derive, and only experience can teach us how to classify so as to get results which are really significant—a possibility of illusion here also! In this freedom, which may easily be abused, we have part of the reason too for the common saying that statistics can be made to prove anything. Further, it is this "concentration of the attention on a small number of artificial groups" which fundamentally differentiates statisticians from historians: for the latter study the conduct of individuals and "endeavour to improve their knowledge of the elements of human nature in much the same way as an astronomer corrects the elements of a planet by comparing its actual position with that deduced from the received elements" (Clerk-Maxwell).

7.21. This preliminary operation of **counting,**[1] especially when the quantities are (as they should be for reliable results) numerous, must be carried out in a systematic way, and the results recorded without dubiety; else confusion enters right at the beginning. Sometimes the counting is done by special machines, but for ordinary purposes the most useful device is to arrange strokes in fives as the class into which the quantity goes is recorded: four parallel strokes are made, and the fifth is a bar crossing these. Another device gives counts by tens—nine dots in a square, which is covered by a cross when the tenth item is recorded. Thus



represents 23. As in all calculations, it is very important to use **checks** here also as often as possible, so as to test if

1 For "Bowley's rules of enumeration" v, Whipple's "Vital Statistics" p. 108.

errors have been made in recording; *e.g.*, a large number of measurements might be taken by 100s, and the total of the classes checked after each hundred is recorded

7.211. For special purposes **special devices** can be adopted. Thus in counting corpuscles in the blood (where the classes are diagnostic,[1] not in this respect quantitative) the objects are practically innumerable, and all that is sought is the percentage number of each of the five kinds of corpuscles that may be present. The counting has to be done for many specimens and it adds much to the labour (especially if a slide rule is not used) and distracts attention if, a percentage has to be worked out when a total of, say, 153 corpuscles has been recorded. A convenient way of avoiding this difficulty is to make the record on ruled squares of 100 small squares. When one of these large squares has been filled, the counting is stopped and the percentage is obtained at once. If counting only 100 blood cells does not give a sufficiently accurate result, the counting is continued for as many more large squares as may be desired, and the totals thus got are divided by the number of these squares used, frequently 5 : cf. **7.5.**

P	P	P	P	P	P	P	P	P	P
P	P	P	P	P	P	P	P		
P	P	P	P	P					
P	P	P	P	P					
P	P	P	P	P					
					M			B	B
					M				
L	L				M				E
L	L	L	L	L	M	M			E
L	L	L	L	L	M	M			E

Fig. 41. Recording Bacteria.

The final counting may be simplified by dividing the square into quarter squares of 25 cells each, and recording the different kinds of objects in different parts of the square according to convenience; or in separate squares if several are

1 An example of what Yule calls "attributes" : Theory of Statistics, Part I.

being used for a count. A square partially filled is shown in figure 41: here the corpuscles are expected in the percentages, P 60 to 65, L 20 to 30, M 10, E 5 and B 0·5 These quantities vary in disease. When a square is filled the only trouble in counting is in the bottom righthand quarter.

But such devices must never be so elaborate as to interfere with the attention the observer has to give to his work; *e.g.*, if any two of the corpuscles were difficult to distinguish, the record should be made by a person other than the observer. (Cf. **4.14** Ex. 2)

7.22. Thus the first stage in dealing with large numbers of observations of one kind is the formation of a **frequency table.** Data are often presented in this form, not as the original individual measurements. The table may be very simple, as in the case of blood-corpuscles, or elaborate according to the complexity of the material dealt with. In classifying examination percentages, for example, it may be doubtful whether the classes should proceed by groups of five or of ten. Very often it is found desirable to have classes of different sizes in different parts of the table, certainly so in statistics of infantile diseases according to age (cf. **7.3** Ex. 5); these diseases attack older people, but so infrequently that it is not worth while showing the number of attacks for each year of age separately: 5 or 10 year intervals suffice. There is also this, that, when we come, as in the next paragraph to plotting the corresponding graph, if the **groups** are **too small** accidental variations (note this specially in rainfall figures) are prominent and destroy the simplicity of the figure. On the other hand, if the groups are **too large,** details are obscured and a sameness comes over the graph—the limit of which would be the horizontal straight line which represents the average.

Ex. 1. Measure the lengths in cms. of thirty or more cucumbers: arrange your measures in frequency tables, using class-intervals 1, 2, 4 and 6 cms. for the respective tables. Examine the maximum girths of the cucumbers in a similar way. Experiment thus with any fairly uniform group of objects, *e,g,,* leaves from one tree.

(Keep in a systematic way the measures you make; they will be useful in connection with **7.6.** The type of result that is got by this process applied to agricultural problems is illustrated in Brunt's "Combination of Observations", pp. 43, 47, where a thorough investigation is given.)

Ex. 2. Form a frequency table (or tables) to analyse the following figures for the June rainfall in Bombay during the 70 years from 1857 to 1926.

(These figures, and those for other months also, may be found in the Times of India Directory, 1926.)

1850								9	14	27
60	22	15	22	23	15	11	13	9	14	26
70	22	9	24	20	19	24	13	35	20	17
80	21	15	28	14	13	5	43	24	16	20
90	25	14	13	21	17	18	28	14	27	21
1900	18	25	10	20	15	7	13	22	15	17
10	24	11	11	26	17	40	24	15	11	23
20	8	27	27	9	7	27	7			

Treat similarly the daily rainfall figures given in **6.52** Ex. 2.

Ex. 3. The accompanying table shows how marks at part of an examination were distributed among candidates at different places where the examination was held. It was pointed out that candidates at A and B had received exceptionally high marks, with presumably the implication that the teaching which candidates resident at A and B received was superior. By re-stating the figures as percentages show that the figures indicate the reverse of this to be true. (Re-grouping of the classes for marks may be useful in bringing out their meaning; *e.g.* 0—59, 60—119, etc.)

Note the slight modes near 20 and 160—indicating respectively the incompetent and the normally brilliant students? (Cf. **7.54**.)

Marks	A	B	C	D
0–	3	2	0	2
10–	7	8	3	0
20–	12	5	4	3
30–	8	10	1	2
40–	13	15	6	6
50–	22	12	5	8
60–	18	33	12	14
70–	24	45	9	11
80–	32	42	11	20
90–	17	25	16	13
100–	13	15	9	7
110–	12	12	4	10
120–	9	8	7	9
130–	1	5	4	5
140–	1	5	0	3
150–	3	3	1	2
160–	5	6	2	0
170–	1	0	0	0
180–	0	1	0	0
190–	1	1	0	0
	202	253	94	115

7.31. NUMBER REPRESENTED BY AREA: The numbers in a frequency table may be dealt with by entirely

arithmetical methods, and in the end these methods are the most satisfactory (**5.3.** Ex. 2 f.n.). But they are sometimes laborious and difficult. Here in the first place we shall consider how the numbers may be represented graphically. We might do this, as in Ex. 4 below, simply by plotting a point above the mid-point of the interval and at a distance from the axis proportional to the number in the interval. But, seeing that the measurements in a class have not all the value which the mid-point of the class represents, it is customary to represent the number of these, not by an ordinate at a point which we regard as typical of the class, but by a column erected on the length of abscissa corresponding to the class-interval. Thus we get a **column-graph** (or **histogram**[1]) in which the area of any column depends on the number in the corresponding class.[2] It can easily be seen that the scale of ordinates is for a column of some standard breadth; if the breadth of the column is 1/5 that of the standard breadth, the height of the column must be made 5 times what it would have been had the column been of the usual breadth.

Ex. 1. Re-state the last sentence, substituting "class interval" for "breadth of column" to get the point of view of **7.22**

Ex. 2. Draw histograms for the frequency tables you have constructed in **7.22** Exs. 1, 2.

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1 Not *historigram*, a word used for the type of time-series considered in **2.311.**

2 Another way of representing the frequency of measurements is the **integral curve**; in it the ordinate represents the sum of all frequencies up to the interval or measurement represented by the abscissa. It is closely related to the *ogive* of **6.31** f.n.: for in it the ordinate represents the measurement below which lie the number of measurements represented by the abscissa; which is equivalent to interchanging "ordinate" and "abscissa" in the definition of the integral curve. If the scales are properly adjusted, the relation between the curves is seen to be that described in **2.11**: you should try to express this relation in words in some particular case, *e.g.*, the heights of the boys in a scout patrol. Cf. also **9.41, 9.44.** The values of the ordinates in the integral curve are got easily from a frequency table by adding on successive frequencies: the final ordinate of course represents the total number of cases, and from it can be drawn readily the parallels whose intersection with the curve give the deciles, etc. Cf. Pearl, "Medical Biometry". p. 119.

Ex. 3. The six sets of exercises given in **4.15** on the slide rule were meant to be of equal difficulty. The results of giving these at random as tests to students of the same class are shown in the accompanying table of the numbers of students gaining, for each of the sets of exercises, the numbers of marks indicated. Sketch six similar column graphs, one under the other, and consider if they indicate that any of the sets are more difficult as tests than the others. In examining the answers it was felt that II was more difficult, while the work of the students who did III seemed inferior. Can we from the figures reject, justify or explain these impressions?

(Before you draw the graphs make up your mind how the tests should be arranged in order of difficulty. The graphs need not contain 16 columns.)

Marks	I	II	III	IV	V	VI	Total
0	1	0	1	0	0	0	2
1	3	1	2	1	2	1	10
2	5	1	1	0	2	2	11
3	4	4	4	4	6	1	23
4	8	5	1	6	3	0	23
5	4	3	2	1	1	6	17
6	3	5	6	1	4	6	25
7	2	4	7	5	5	5	28
8	1	2	7	6	4	8	28
9	4	6	4	5	4	2	25
10	3	3	4	4	3	7	24
11	4	6	2	1	6	5	24
12	5	3	2	3	1	0	14
13	0	1	3	7	5	4	20
14	1	2	2	3	2	1	11
15	0	1	1	0	2	0	4
	48	47	49	47	50	48	289

Note: The other main factor that may cause differences in the results is the varying *ability of the students*: the fact of this variation is apparent in the graphs. But though it varies differently in each set of students, it may be *taken as constant* for a whole set (for the assignment of student to test was entirely random—an equal number of copies of each test, these placed in any order, the students taken without selection and assigned a test in the order they chose to come); then the criterion of the easiness of each test is the total area of another graph which represents, not the number of mark-earners, but their mark-earning capacity. This graph could be constructed, but it has an easily worked arithmetical equivalent. (Strictly we should take the *average* mark-earning capacity, for the numbers of mark-earners in the groups are not quite the same: cf. p. 118.)

Conversely, if we *take the difficulty of the tests as constant*, we get from this second graph a direct measure of aggregate ability.

This seems very confusing. But it shows the need for cultivating a balanced judgment in order to approximate to the truth.[1] Cf. **6.15, 6.411.**

Ex. 4. In figure 42 are shown two distributions of nebulæ with respect to the Milky Way (or Galaxy). Make from the graphs frequency tables giving the numbers of nebulæ found at different distances north and south of the Milky Way. Describe the nature of the distributions, and state how far they are consistent.

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1 For a similar balancing of assumptions as to uniformity, either of death-rate, or of increase of population, see Whipple's "Vital Statistics", pp. 263, 264.

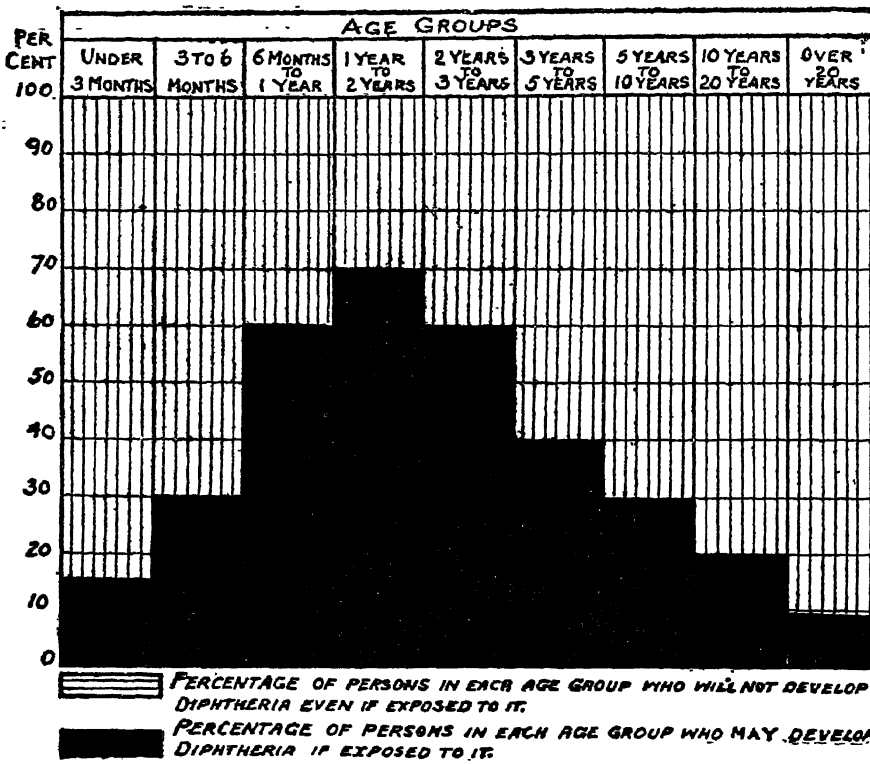

Fig. 43. Diphtheria-risks in certain age-periods.

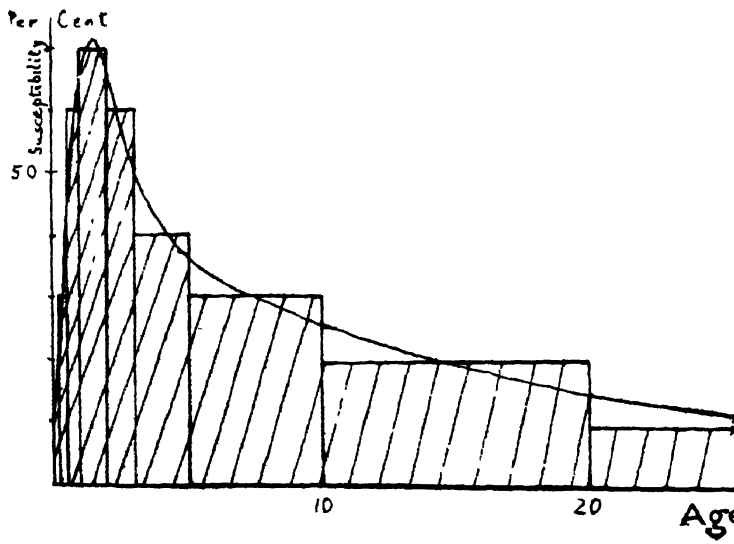

Fig. 44. Diphtheria-risks at any age.

(This is an example of how observations which can be made only by the most skilled observers with special photographic and other equipment can be presented in a very simple way. A complete survey of the sky for this purpose could not be undertaken, but 139 selected areas were examined thoroughly. To avoid accidental bias these areas were taken alternately and arranged in two series of groups, each of five areas (save one in which there were only four areas); for each of these groups the sum and the mean latitude of the nebulæ were found, and plotted as in figure 42. It is evident that there is considerable general agreement between the series, and this gives the hope that further research along these lines will be fruitful in revealing some definite law of distribution; this, however, will be complicated by the fact that there is a band, in which nebulae are frequent, crossing the Milky Way almost at right angles. Note in the graph that knowledge of the southern hemisphere is defective. The original figures may be found in the Astrophysical Journal **62** 171.)

Compare the results got from the graph with the figures got by treating the areas as a single series of groups of five areas each (save one): devise other ways of grouping the figures for graphical representation.

Mean galactic	N	78	69	62		58	52	48	43	40		36	31	28	24		19	14	12	6	0
latitude.	S		69				54		45				38	31	25	20			14	12	6	0
Numbers		232	63	54		75	48	47	33	72		69	55	18	12		7	4	0	2	0
of nebulæ			68				31		32				20	59	25	1			1	4	0	0

Ex. 5. Draw a histogram to represent the distribution with respect to age of the death-rate due to diphtheria in England and Wales from 1891 to 1900, as given in the following table. (From Yule's "Statistics", p. 98). Cf. **9.3** Ex. 4 Col. e.

Age.	No. of deaths	No. for year of age
0-	4,186	4,186
1-	10,491	10,491
2-	11,218	11,218
3-	12,390	12,390
4-	11,194	11,194
5-	23,348	4,670
10-	4,092	818
15-	1,123	225
20-	585	117
25-	786	79
35-	512	51
45-	324	32
55-	260	26
65-	127	13
75-	35	?
	80,671	

(You may easily express by the slide rule the numbers in the last column as percentages of the total deaths before plotting them: but the diagram is more informing if the original numbers are shown on the vertical scale. A percentage scale may with advantage be shown in addition to the actual numbers.

The frequencies are almost always marked along the vertical scale. I have seen in a medical journal a graph in which along the vertical axis was shown the average number of parasites observed under certain conditions, while along the horizontal axis was recorded the number of observations of each such number; but I remember seeing no other direct breach of this convention.)

7.311. Difficulty arises when *at either end of a frequency table* a class is given, described either as "below or above" a certain bounding value. Only knowledge of the circumstances of measurement, etc., can enable us to judge how broad the corresponding columns should be. It is often very important to have exact information about the extreme classes. This is illustrated in **7.312**, the examples of **7.32**, Ex. 1, etc.

7.312. In figure 43 is reproduced from a weekly report, dated August 18, 1922, issued by the United States Public Health Service, a "chart" which looks like a frequency diagram. In reality it belongs to the category referred to in **7.11**, for the *time-scale* is *not uniform*; also the height of a column is a percentage of the total in the corresponding age-class, not of the universe considered, as in an ordinary frequency graph. The diagram is published with much simple information about diphtheria, and is probably meant for popular instruction. But had its arrangement been more strictly scientific, it might have achieved its purpose more effectively. It may very reasonably be held that *representation of risks* are most truly given by comparison with the number between given ages who are exposed to that risk. But in the figure that is constructed so it should be clear that there is a changing scale—the columns should not be contiguous, as they may be when the unit of area is the same in successive columns.

The superscription on the chart is "Showing the early increase in susceptibility to diphtheria, followed by the development of a natural immunity." In figure 44 the essential part of the diagram is re-drawn: the change in the aspect of the diagram is most striking. The very early rapid rise in risk was to be expected, but few would have predicted the very considerable risk that persists up to 20 years of age. The latter impression, however, is modified when the distribution curve is substituted for the histogram: also the columns to the right are proportionally higher than if they represented percentages of the whole population. (Cf. Ex. 5.)

These diagrams should be compared with that got from Ex. 5; this histogram is given in Yule's "Statistics",[6] p. 97. The English figures are for deaths, not attacks; hence the skewness of the curve is even more marked than in the American diagram. The American figures are apparently estimates, for they are mostly in tens. Precise figures are given in Whipple's "Vital Statistics" pp. 376, 377.

7.32. GENERALISING THE GRAPH: DISTRIBUTION CURVES: It is frequently desirable to replace the column-graph by a continuous curve. In some ways this is a more truthful representation of the facts; for all the measurements within a group are not alike, and are not sharply contrasted with the members of the groups below and above it as the

stepwise representation seems to imply; the continuous curve removes these misrepresentations, though there is no direct guarantee that it is more correct than the other.

The word "corrrct" is inappropriate here. The data we select, or are given, are but a sample of the measurements the nature of which we are discussing; and experience shows that the more numerous the data are, the closer to the frequency curve does the shape of the column graph become. What we are really doing in smoothing the graph is to suppose that the number of cases has become innumerable while the class interval has become infinitesimal, *i.e.*, we are proceeding to a limit (cf. **3.2**). But in going to a limit we must keep checks on ourselves, else we get nowhere; the restriction we have to observe is what we must pay attention to now.

The important consideration is that the numbers in the column graph are represented by areas, and therefore the continuous frequency curve should have **the same area.** There is no more rigid rule than this for drawing the continuous frequency curve, and only practice can settle how best to draw the curve in each case so that the area between the curve, the axis and the bounding ordinates for a class is equal to the area of the corresponding column.[1]

Ex. 1. Represent the following figures (copied from Bombay Government Reports) in a graphical form. Are the frequency graphs thus obtained suitable for conversion into distribution curves? (cf. also Census of India, 1921 IX xviii).

(a) Percentages of occupants of tenements living in a certain number of rooms. (Working Class Budgets, page 24)

No. of rooms	1	2	3	4	5	6 and over
Bombay	66	14	8	5	4	3
London	6	15	20	17	11	25 (Note the defect in these figures.)

(b) Percentages of cotton operatives whose daily wage lies between specified limits (Cotton Industry, page 6)

Wages in annas:	12	12–18	18–24	24–36	36–48	48 upwards
Bombay	4·8	35·3	23·8	27·2	5·2	3·7
Ahmedabad	10·1	36·8	20·0	26·9	5·6	2·6
Sholapur	49·5	23·9	14·2	10·3	1·3	·8

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1 In figure 44 no such clear rule can strictly be followed, for the numbers in each age-group are not equal (cf. Ex. 4); and so the actual number, corresponding to the percentage number, of diseased people represented by unit area varies from column to column. In drawing the smooth curve general considerations of *continuity in variation* can in this case be the only guide.

Ex. 2. A class of students, who had just escaped the fetters of school, was divided into two sections A and B according to ability as shown by examination marks. Some weeks later they were tested with the result shown in the accompanying table. Represent these results in one graph, and suggest an explanation of the fact that the graph for A is double-peaked. Cf. **7.22** Ex. 3 ; also **7.54**.

Marks	A	B
0-	3	26
5-	14	41
10-	13	31
15-	11	22
20-	19	9
25-	10	7
30-	8	5
35-	7	1
40-	10	0
50-	3	0
60-	0	0
70-	2	0

Ex. 3. Plot the information given in the accompanying table of *fluctuations* (cf. **6.4**) of daily average spot prices (0·01 dollar) of cotton at ten places from the prices given by a moving average (**2.311**) of five days. Draw the curve which shows the distribution of these fluctuations from their general trend. (Cf. Moore, "Forecasting the Yield and the Price of Cotton", p. 26: Macmillan. This curve is a close approximation to the symmetrical frequency curve which is got when simple measures like height of men of one race are classified: cf. **7.51** ; it is called also the **normal error curve**, a name which is appropriate in this example.)

Fluctuation	Freq^y.
−·165-	3
−·135-	3
−·105-	4
−·075-	23
−·045-	55
−·015-	107
·015-	54
·045-	16
·075-	7
·105-	2
·135-·165	1

Ex. 4. Exhibit on one graph the distributions by age of the sections of the populations of Bombay City given in the following table. (Census of India, 1921, IX xi, xii.)

Do you see anywhere in these statistics a tendency for people to give their ages in round figures? This may be more marked in other cities than in Bombay: test this suggestion from other figures given in the Census Reports.

	Hindus	Jain	Parsi	Musalman	Christian	Total	Cf. Sholapur
0-	9,060	241	767	1,607	804	12,662	3,526
1-	5,640	135	425	1,313	482	8,075	1,776
2-	10,543	275	575	2,407	888	14,852	2,632
3-	11,738	293	820	2,572	888	16,500	2,447
4-	11,733	246	831	2,670	872	16,543	2,970
5-	63,348	1,379	4,226	13,769	4,422	88,032	14,192
10-	63,170	2,250	4,647	14,937	5,189	91,383	13,128
15-	78,640	3,287	4,614	15,921	6,965	110,281	9,812
20-	121,464	4,029	5,327	23,817	10,487	166,242	10,873
25-	136,692	3,691	5,122	27,005	9,773	183,483	11,440
30-	118,661	2,980	5,179	25,990	8,164	162,093	10,749
35-	74,555	1,983	4,357	17,274	6,343	105,213	6,532
40-	55,242	1,356	4,239	14,482	4,742	80,683	7,449
45-	26,797	692	2,946	6,581	3,099	40,431	3,524
50-	25,300	562	3,087	6,998	2,443	38,708	5,230
55-	7,704	175	1,669	1,963	996	12,636	1,595
60-	11,283	192	1,649	3,503	928	17,733	3,785
65-	2,297	45	780	604	327	4,096	679
70-	3,523	73	974	1,272	357	6,268	1,592
Total	837,690	23,884	52,234	184,685	68,169	1,175,914	113,931

(For the purpose stated you will have to convert by the slide rule the numbers given above into percentages of their respective sections. You cannot expect, of course, to represent on a graph the fourth, fifth and sixth figures in numbers; but these, though they look as important in the table as other figures, are not really important. The graph does represent the important figures.

If you find the percentages very much alike and if colours are not available, you will have to represent the facts on separate diagrams: but this should be avoided if possible. These percentages may also be combined so as to show to what type of population, according to the accompanying classification, each community given in the above table should be assigned: *v.* Whipple, *op. cit.* p. 189: cf. Census of India, 1921 VIII 87).

Sundbärg's **Types of Populations.**

Age-group	Progressive	Stationary	Regressive
0-	40	33	20
15-	50	50	50
50-	10	17	30

Ex. 5. Draw graphs for the extreme cases of frequency distribution of diameters of blood cells of individuals within each of the three classes given on p. 127. Describe how these vary from the forms of the curve for the aggregate numbers in each class (**7.5**). Cf. **7.54**.

Ex. 6. In the Census of India, 1921 VIII 126, ten distribution curves are given of married people by age. By counting squares test these for equal areas. Find as in **7.4** typical numbers for these distributions.

7.41. CALCULATION OF TYPICAL NUMBERS. To find the **arithmetic mean** of the measures arranged in a frequency distribution, we suppose that all those in a class have as their mean the middle value of the class. This is not usually true, but uneven distribution in one direction in the classes towards one side of the corresponding graph will tend to be balanced by unequal distribution in the opposite direction on the other side, if the figure be a simple one with one mode: and the mean value calculated for the whole will not be far from the actual mean value which would be got if all the measures were added.

We can **make the calculation much easier** by taking the differences with proper sign between the mean class-values and a value we judge to be near the mean of the whole. These *differences multiplied by the class frequencies, added,* and *divided by the total number of measurements,* give the difference between the calculated A.M. and the chosen value. We can convince ourselves of the correctness of this procedure by simple examples. Dealing first with *direct measures*: the

mean of the following nine numbers was badly judged to be 14, and yet the result of the calculation by the method indicated is exactly that got by the ordinary method.

Nos. 9 11 13 12 16 17 8 14 11; total 111(=9×12·33)

Dces. (from 14) $\left\{\begin{matrix}-5 & -3 & -1 & -2 & & & -6 & & -3; & \text{„} & -20 \\ & & & & 2 & 3 & & & & & +5\end{matrix}\right\}$ $\begin{matrix}-15 \\ =9\times-1\cdot67\end{matrix}$

$\therefore$ the A. M. $=14-1\cdot67=12\cdot33$.

So also for a simple *frequency table* where we represent the classes by successive ordinals; we can always do this if we *make the classes uniform* and then insert the value of the unit, $\frac{1}{4}$, 5, or whatever it may be, at the end of the calculation thus simplified:

Class	8	9	10	11	12	13	14	15	16	17	
Freq.	1	1	2	4	5	8	12	16	14	6	Total 69
Dces. (from 15)	−7	−6	−5	−4	−3	−2	−1		1	2	
Prods.	−7	−6	−10	−16	−15	−16	−12		14	12	$-56\div69=-\cdot811$, by slide rule.

The A. M. is thus found to be 14·189. Checking by direct calculation we get $979\div69$, which by the slide rule is 14·18: the class-interval is the unit here.

Ex. 1. Calculate the mean of the June rainfall in Bombay from the frequency tables you have constructed (**7.22** Ex. 2).

(The value, as given in the Indian Year Book, 1924, p. 282, is 20·56 inches. This is doubtless the A.M. of the actual falls recorded to date, not that calculated from any frequency table: it may also include records prior to 1857.)

Ex. 2. Verify that the mean fluctuation in **7.32** Ex. 3 is −0·002. (Moore, *loc. cit*, p. 21).

Ex. 3. Use the slide rule to check the values of the variability (*i.e.* the ratio *S. D.*: *Mean* as a percentage) given in the following tables, which give the individual typical numbers in two of the sets of **7.5**: verify also the means.

I	Mean	S. D.	Var$y$	II	Mean	S. D.	Var$y$		Mean	S. D.	Var$y$
	6·319	0·79	12·5		7·339	0·47	6·4		7·200	0·45	6·2
	7·225	0·69	9·5		7·443	0·47	6·3		7·274	0·52	7·1
	6·900	0·70	10·1		7·302	0·52	7·1		7·487	0·44	5·8
	6·611	0·57	8·6		7·280	0·50	6·9		7·283	0·41	5·6
5	7·179	0·92	12·8	5	7·124	0·48	6·7	15	7·307	0·43	5·8
	6·891	0·76	11·0		7·211	0·46	6·3		7·311	0·47	6·4
	7·215	0·61	8·4		7·186	0·52	7·2		7·169	0·48	6·6
	6·943	0·51	7·4		7·091	0·46	6·4		7·280	0·44	6·0
	6·702	0·66	9·8		7·114	0·47	6·6		7·160	0·49	6·8
10	6·813	0·65	9·5	10	6·968	0·46	6·6	20	7·231	0·47	6·5
Mean values								Mean values			
	6·879	0·686	9·98						7·239	0·470	6·46

Ex. 4. Check the values of the mean given on p. 127 for the frequency tables showing the diameters of blood cells of individuals, both healthy and diseased. (In each case the frequencies should total 500.)

7.42. The calculation, from a frequency table containing a considerable number of measures, of the typical number we have called the **median** is easy[1] compared with that for the A.M., though in this calculation we make the same assumption of even distribution in a class-interval.[2] We first by simple addition find *the class-interval in which the median must be; then* by proportional parts (**1.4**) we find *the position in this interval* of the central measurement for the whole, assuming, as has been already said, that the successive measurements change evenly in the interval. In the simple

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1 This case of calculation, however, must not be taken as marking the median as a superior typical number: sometimes it becomes rather unintelligible. If you are interested in this point, see Yule's "Theory of Statistics", p. 119.

2 This assumption is made also in order to find for the *mode* a definite place within a class interval: cf. King's "Statistical Method", p. 124. If the mode occur in an interval of breadth c, beginning at l, and is flanked, as shown in figure 45, by intervals of frequencies f_1 and f_2 then the position of the mode is $l + \frac{f_2}{f_1+f_2}c$. Thus the position is fixed, not immediately by the number within the group as in the case of the median, but by numbers in the neighbouring groups: if circumstances should render it desirable (**7.22**), f_1 and f_2 could be made to include more frequencies than those immediately adjacent. (The same assumption can be used in calculating the value of the *mean deviation*, **6.12**, of the distribution given in a frequency-table.)

example of a frequency table we took in **7.41** the middle measure is the 35th.: this occurs in the class marked 15; for the sum of the frequencies in the groups from 8 to 14 is 33. The position of the 35th. measure in group 15, in which there are 16 measures in all, is given by $\frac{2}{16}$ × the class-interval.

If this interval is taken to begin at 15, the median value is 15·125; if it ends at 15, the median is 14·125: this point as to the classification must be made clear when the table is given. In calculating the A.M. we implicitly took 15 as the *middle* of the class-interval: with this assumption the interval begins at 14·5 and the median is then 14·625. This is the number to compare with the mean 14·19: we see that the difference is rather large, but it is in the direction that is always found for unsymmetrical (or skew) frequency curves: the mean is nearer the "long tail" of the figure than the median. (Yule, "Theory of Statistics"[6], p. 114)

The positions of the lower and upper **quartiles** are determined in an exactly analogous manner. In the same example, the lower quartile is between the 17th. and 18th. measures, and so it is in the group marked 13. Its distance from the beginning of this interval is $4\frac{1}{2} \div 8$, *i.e.*, 0·563 of a unit; this can be interpreted as above for the median. So also for the upper quartile.

All these numbers will be exemplified in **7.5**. They should also be calculated for the rainfall figures and for other frequency tables given in **7.2, 7.3,** and the results compared. They offer an effective and concise way of comparing sets of figures like those in **7.22** Ex. 3, **7.31** Ex. 3, and **7.32** Ex. 4.

Students should organise themselves to work in pairs at given sets of statistics, and then compare results for different types of distributions. These are classified in an easy and interesting way in Yule's "Statistics" Chap. VI, Nunn's "Algebra" II p. 438, and in other places; but we shall not wait over this classification—you can discover these types for yourselves.

Ex. 1. The distribution of annual rainfall at Mahableshwar from 1829 to 1915 is given in the accompanying table of frequencies in classes of 10 ins. of rain, beginning with 130–9 and ending with 400–10: find the mean annual rainfall during this period and the mean deviation. Compare these with the median value and the quartile deviation.

100 ins.	1, 0,	0, 1, 1, 2, 1
200	3, 3, 6 5, 9	6, 13, 7, 8, 6
300	2, 4, 2, 2, 3	0, 0, 1, 0, 0
400	1	

Ex. 2. A class of 234 students is grouped according to marks, 7 falling in a group from 52 to 59, and the others in the successive groups of 10 marks up to 140–9 thus: 12, 20, 50, 42, 40, 31, 22, 9, 1. If it is desired to divide the class into six sections according to ability, what marks will distinguish approximately between the sections if the numbers in them, beginning with the best section, are fixed to be 120, 30, 25, 25, 19, 15?

7.5. A STUDY OF ANÆMIAS:[1] For twenty healthy people, for twenty patients suffering from pernicious anæmia, and for ten from anæmia after hemorrhage measures were made of the size of the blood cells, 500 cells being measured in each case. The results of these measures may be grouped as in the frequency tables on p. 127. The unit of measurement used, μ, is 0·001 mm., but that does not concern us; we can take conveniently the figure that designates a class as the lower boundary of that class, though all that is stated in the original article is that the diameters are measured to $0{\cdot}25\mu$. When you draw the graphs for *the grouped figures* in these frequency tables, you will see very clearly the general differences between conditions of bloodcells in the two kinds of anæmia and in healthy persons.

Ex. Draw the graphs and describe their characteristics. Try to suggest a meaning for these. (In the original paper all graphs are reduced to a total of 500 cells (cf. **7.31**), which is convenient when there are comparisons to be made, as in the original paper, with graphs for individual cases. For this exercise only it is slightly less trouble to work with a total of 10,000 in each case, but the above adjustment should be made for the sake of comparison with **7.32** Ex. 5.)

7.51. To the right of the columns of grouped counts of cells are shown in small type the totals of classes up to those in which the lower quartiles, medians, and upper quartiles occur. The calculations of these numbers for the distribution of the sizes of healthy cells are respectively

$$7{\cdot}00+\frac{2500-2236}{1832}\times 0{\cdot}25=7{\cdot}036;\quad 7{\cdot}25+\frac{932}{2060}\times\tfrac{1}{4}=7{\cdot}363;$$

$$7{\cdot}50+\frac{1422}{1784}\times\tfrac{1}{4}=7{\cdot}700$$

1 From the Journal of Pathology and Bacteriology **25** 487, 1922 Oct. "Diameters of Red Cells in Anæmias." by Cecil Price-Jones.

Thus the median diameter is considerably larger than the mean, given in the paper, 7·210.

−		+	
$2 \times 2{\cdot}25$	4·5	$2060 \times {\cdot}25$	515
$3 \times 1{\cdot}5$	4·5	$1784 \times {\cdot}5$	892
$25 \times 1{\cdot}25$	31·25	1135×75	851·25
92×1	92	620×1	620
$259 \times {\cdot}75$	194·25	$236 \times 1{\cdot}25$	295
$613 \times {\cdot}5$	306·5	$74 \times 1{\cdot}5$	111
$1242 \times {\cdot}25$	310·5	$18 \times 1{\cdot}75$	31·5
		4×2	8
	−943·5	$1 \times 2{\cdot}5$	2·5
	2382·75		
	3326·25		+3326·25

This discrepancy led to suspicion as to the accuracy of the value, 7·210. The A.M., for the cells is recalculated here as shown, 7 being taken as the measure from which departures are reckoned. The result is an average departure of 0·238275, which differs from the ·210 given in the paper. To compare this with the median value just obtained we note that there 7 was taken as the lower limit of the interval, while in calculating the A. M. 7 was the middle value of an interval, 0·125 lower in the scale. Accordingly we must add 0·125 to the A. M. just obtained, 7·238, and we get 7·363, exactly the value we found for the median. In view of the symmetry of this distribution the result is not surprising (cf also fig. 47, which is for only one case : the grouped numbers would give points even more closely on the straight line). In checking similarly the other calculations, remember the remark in **7.42** about the relative position of typical numbers in an unsymmetrical distribution.

In what follows we assume that the typical numbers shown in the table on p. 127 are correct : this is of importance only when doctors apply these results, and does not affect the principles we have to study.

7.52. Dr. Price-Jones uses to characterise the distributions quantities we are not to discuss : these are given below their respective columns in order that we may compare them with the easier characteristic numbers which we already know. "S.D." means "**Standard deviation**", a very frequently used measure of "scatter", *i.e.*, of closeness of the measures generally to the typical number (cf. **6.12**) : with this, given as 0·45, should be compared the quartile deviation, $\frac{1}{2}(7{\cdot}700 - 7{\cdot}036) = 0{\cdot}332$.

The **variability** or **coefficient of variation** is here taken as the percentage ratio of the standard deviation to the arithmetic mean, the two numbers which precede it in the table : it may be regarded as the ratio of any measure of deviation to any value typical of the absolute size of the group of numbers; and so it gives a measure of the **relative scatter** of a curve; it is appropriately a mere number, and independent of the units used.

Ex. 1. Calculate for the other two grouped columns the medians and quartiles, and obtain for all three the coefficient of variation as the percentage ratio of the quartile deviation to the median value.

Ex. 2. Check the value of the A. M. given for each distribution of the sizes of all the cells within a group.

Ex. 3. "The rough rule that the semi-interquartile range is usually some 2/3 of the standard deviation : it is strictly true for the normal curve only." (Yule, *op. cit.* p. 310. : cf. also p. 146, where is given the approximate rule for nearly symmetrical distributions that the *mean deviation*, **6.12**, is 0·8 times the standard deviation.) Test this statement by what you calculate from Table II.

7.53. Dr. Price-Jones discounts any interpretation of the figures for the *grouped* individual *cells*, for the material measured (the blood cells) is so heterogeneous; apart from differences between individual persons, the sizes of cells vary with exercise, time of day, etc. (It is very important to deal with material that has not been disturbed by chance influences[1]; in this case the specimens were all taken at the same time of day.) Accordingly he takes for each case (*i.e.*, person!) the **mean value** of the diameters and the coefficient of variation, and summarises his results for *grouped cases* in the way given at the foot of p. 127. (The details for two of the groups are given in **7.41** Ex. 3: I, post-hemorrhage anæmia; II, healthy blood.) This summary is interpreted as follows:

"It appears (from the summary for cases on p. 127).

(*a*) That the mean diameter of the red cells in pernicious anæmia is greater than the mean diameter of the red cells in healthy persons. The smallest mean diameter of the pernicious anæmia cases is equal to the largest mean diameter of the healthy persons, but otherwise they do not overlap, or, in other words, all the mean diameters of the pernicious anæmia cases are greater than those of the healthy persons excepting one.

(*b*) That the mean diameter of the red cells in the cases of anæmia following hemorrhage is smaller than the mean

1. Cf. the emphasis on *basal* metabolism in **9.52**, i.e., metabolism during absolute muscular repose in the morning, 12 or 14 hours after food has been taken.

diameter of the red cells in healthy persons. The biggest mean diameter is less than the average mean diameter of the healthy persons. In seven out of ten of the cases the mean diameter is less than the smallest mean diameter of the healthy persons.[1]

(*c*) The mean coefficient of variation of the red cells of the pernicious anæmia cases is more than twice the mean

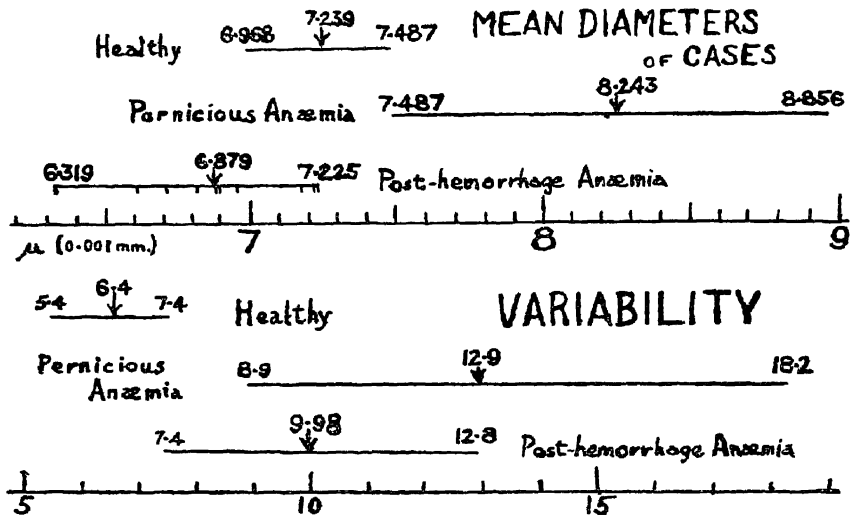

Fig. 46. Comparison of Ranges of Values.

coefficient of variation of the healthy persons, and the smallest coefficient of variation of these cases is greater than the biggest coefficient of variation of the healthy persons.

(*d*) The mean coefficient of variation of the red cells of the hemorrhage anæmia cases is half as much again greater than the mean coefficient of variation of the red cells of the healthy persons, and the smallest coefficient of variation is equal to the biggest coefficient of variation of the healthy persons."

1 This sentence can be verified from the figures given in **7.41**, Ex. 3 I.

7.531. Though the matter is not relevant to the representation of frequency distributions, it may be remarked here that this summary of results may be made much clearer by graphical representation such as is referred to in **2.33** : this is shown in figure 46.

If phrases like "excepting one" at the end of (*a*) are not specially important in the summary, the figure may be relieved of the burden of numerical values attached to the arrows, etc., and become so much the more striking through direct dependence on the two scales shown : there is an advantage in using graph paper in this case. An arrow indicates the mean in a range of values. But the graph may be adapted with ease to represent fully even such a statement as the last sentence of (*b*). This is done by putting marks on the post-hemorrhage line in the positions corresponding to the mean diameter for each patient.

The positions of the means of mean diameters in the upper half of the ranges corresponds in some degree to the skewness of the respective frequency curves. But the very distinct departures of the mean variabilities from the middle of their range would not have been foreseen so easily; and so this graphical representation has the additional advantage over the verbal statement of making clear unsuspected facts, which may or may not have medical significance. In this connection there may be a real advantage in representing each case in its proper position on its line, as has just been suggested.

7.54. At **7.41**, Exs. 3, 4 refer to these frequency tables. Individual cases are given on p. 127, each the extremes of their class ; they show clearly the greater range of cell-diameters in pernicious anæmia patients than in healthy persons. It should also be noted that in Case No. 5 the distribution shows *several modes* (**6.141**) : this is common in pernicious anaemia (cf. case 13), and has led to the surmise that in this disease there are *cells of three different kinds* present in the blood ; cf. **7.22** Ex. 3. This has still to be verified, for the measurement of diameters was so laborious a process that the investigation had to be left incomplete.

7.55. In connection with functional scales it is very instructive to study the greatly reduced figures 47 and 48 where striking use is made along the *y* axis of a scale which is entitled "Percentage of red cells less than the

stated diameter", the diameter being marked on a uniform scale along the x axis. (The original numbers plotted in these diagrams are given in Table II on the right of their respective classes.) The y axis scale is closely connected with the **"normal error" curve** (cf. **7.32** Ex. 3); but we are not to deal with that in this book. Its connection with the table[1] given on p. 310 of Yule's "Theory of Statistics" may however be easily seen. Starting from

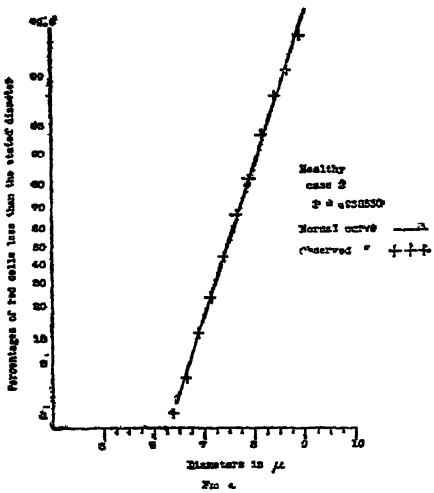

Fig. 47.

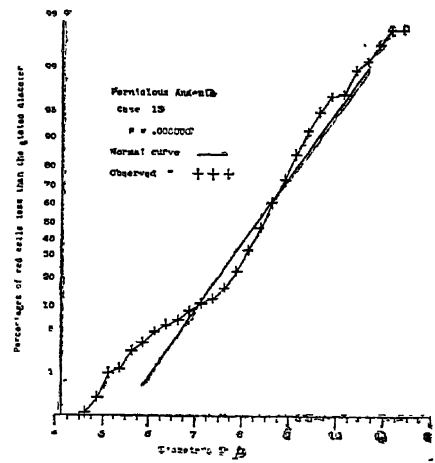

Fig. 48.

Probability Ruling. (The percentages plotted are easily obtained from figures given in Table II, Nos. 2, 13.)

graduation 50, the second column of the table on p. 128 shows in cms. the distances of the graduations of the y scales on the original diagrams. The third column gives the corresponding area-fractions (cf. **7.3**) from Yule's table; it is evident that the ratio of corresponding numbers in the second and third columns is almost constant. Hence

1 This table is given in some form in all books on statistics, e.g. Jones 284, Bowley[4] 271, Pearl 362. Paper with the ruling in one direction based on it is called *probability paper* (Whipple *loc. cit.* 451).

TABLE II—Distribution of the sizes of red cells in 50 people

Diam.	Healthy blood				Pernicious anæmia				Post-hemorrhage anæmia			
μ	No. 14	20 persons		No. 20 No. 2	No. 17	20 patients		No. 5 No. 13	No. 8	10 patients		No. 5
3·50										2		
·75						2		1		2		
4						6		1		3		
·25						11		1		9		
·50						16		3 1	3	7		
·75		2		2		26		2 1	1	24		
5		0		0		48		2 4	1	23		1
·25		0		0		51		3 1	0	49		3
·50		3		0	6	89		8 6	1	97		5
·75		25		2 2	1	99		6 4	1	189		5
6	1	92		4 0	1	118		3 7	15	350		25
·25	5	259		4 3	3	166		4 5	30	435	1190	44
·50	20	613	2236	31 12	4	218		15 6	58	661	1851	54
·75	55	1242		59 39	11	312		11 9	102	731		61
7	96	1832	4068	96 63	12	360		21 10	104	721	3303	77
·25	110	2060	6128	114 100	39	473	1995	19 8	90	601		61
·50	118	1784		86 112	52	632		26 17	54	462		47
·75	59	1135		68 79	71	729		19 33	26	269		33
8	24	620		22 57	77	948	4304	33 57	10	177		22
·25	10	236		11 23	80	998		35 64	3	88		16
·50	2	74		1 6	56	1003		44 75	0	28		4
·75		18		3	41	878	7183	46 60	1	28		12
9		4		1	18	760		34 52		12		7
·25		0			13	573		30 37		11		8
·50		1			5	485		26 18		7		3
·75					5	300		17 10		5		4
10					1	255		24 1		4		3
·25					3	135		19 8		1		1
·50					1	119		15 2		0		0
·75						69		8 2		2		2
11						60		6 1		2		2
·25						25		9 0				
·50						18		5 1				
·75						10		2				
12						5		2				
·25						3						
		10,000				10,000				5,000		

Diameters					*Cells*						
Min.	6·00	**4·75**	4·75	5·75	5·50	**3·75**	3·75	4·50	4·50	**3·50**	5·00
Mean	7·283	**7·210**	7·231	7·443	8·022	**8·243**	8·512	8·314	6·943	**6·850**	7·179
Max.	8·50	**9·50**	8·50	9·00	10·50	**12·25**	12·00	11·50	8·75	**11·00**	11·00
S. D.	0·41	**0·45**	0·47	0·47	0·72	**1·15**	1·45	0·99	0·51	**0·75**	0·92
Coefficient of Variation											
	5·6	**6·2**	6·5	6·3	8·9	**13·9**	17·0	11·9	7·4	**10·9**	12·8
Mean Diameters					*Cases*						
Min.		*6·968*				*7·487*				*6·319*	
Mean		*7·239*				*8·243*				*6·879*	
Max.		*7·487*				*8·956*				*7·225*	
Variability : (Coefficient of Variation)											
Min.		*5·4*				*8·9*				*7·4*	
Mean		*6·46*				*12·9*				*9·98*	
Max.		*7·4*				*18·2*				*12·8*	
Mean S. D.		*0·470*				*1·07*				*0·686*	

the scale used can easily be constructed from Yule's table. Cf. **2.22.**

Graduation	Distance	Table
50	0	0
60	0·52	0·26
70	1·12	0·51
80	1·85	0·86
90	2·70	1·3
99	4·90	2·35
99·9	6·4	3·09

What we have to note here is that the use of this functional scale along one axis results in the facts about a *normal* (i.e. healthy) individual being represented along a straight line.

Ex. From the measurements given in Yule's table construct figures such as 47 and 48 to represent the facts given in the other frequency tables in Table II. Modify the results you have obtained in **7.22** Ex. 1 so as to test whether they give points on a straight line when plotted on this probability ruling.

7.61. MULTIPLE CORRELATION. On several occasions we have used the term *correlation*[1] in referring to a relation of agreement between two quantities (**6.41, 6.511**); but the word is frequently used in a restricted sense with reference to a **relation between frequencies** which may be found to exist between pairs of measurable characteristics, *e.g.*, overcrowding and infant mortality. From this point of view correlation appears as a natural extension of what we have done hitherto in this chapter: this has been simply to fix intervals along the x axis, and in the columns corresponding to these to note each occurrence of a measurement within that interval; the numbers of occurrences we have taken as y ordinates and considered the properties of the frequency curves thus obtained. Similarly, if we know two measurements, say, weight and height of each individual, we can mark convenient intervals of these quantities along the x and the y axes, and by parallel rulings get rectangular compartments, each corresponding to one interval on the x axis and another on the y axis. It is not difficult, in classifying the measures for individuals, to record in which compartment each pair of measures should be placed. If we wish to treat

1 A footnote in "Mental Measurement", p. 121, indicates that the word "correlation" is much sought after for various purposes! Does the fact that in **6.52** Ex. 2 correlation of some degree was found, however the sets of numbers were adjusted, help to explain the tendency to vagueness in the use of this word?

these frequencies of pairs of measures as we dealt with the frequencies of single measures, we can regard them as represented by columns or lines drawn from the respective compartments perpendicular to the *xy* plane; and then the tops of these may be taken as lying on a *frequency surface* (cf. **7.32**): number in this case would be represented by volume. But our first concern here is with the **counting** (**7.21**).

Ex. Arrange pairs of measurements for individuals such as you made in **7.22** Ex. 1 in the way described above.

This method we can extend to three measurable associated variables *e.g.*, height, weight, and chest-measurement. In such a case the compartments are cuboids bounded by parallel planes through the ends of intervals marked on the three axes. It is not easy now to picture how the numbers for which each of these triplets of measures occur should be represented: we have almost reached the limit of our powers of graphical representation; but we have by no means reached the limit of the powers of mathematical method. The methods which are explained in books on statistics for elucidating the meaning of the frequencies of pairs can be extended to the frequencies of any number of measures of similar individuals or circumstances. The relations and properties that are found become, of course, more and more complex and difficult to appreciate; but you now know that you can apply to the mathematician for help, if you have to set straight a tangle of many inter-relations in a great number of like cases: only make your problem very definite and clear to yourself before you can expect him to help; for the mathematician is very exacting about definitions (**1.8**)!

Ex. Is the triangular diagram of **9.5** of any utility in making the counts of measures in threes?

7.621. CORRELATION. "The full significance of correlation is only to be realised after a careful study of the general theory of correlation of numerous variables, of which the correlation of two variables, measured by the correlation coefficient *r* (cf. **6.41**), is only a particular case."[1]

1 Brown and Thomson, "Mental Measurement", p. 146.

Bearing this warning in mind, and remembering that all that we are attempting here is to see some little distance into the possibility of applying mathematical methods, we may glance at the initial stages in the arrangement of counts of pairs of measurements. If we have counted the lengths and maximum breadths of many leaves from the same tree into the rectangular compartments of what is called a *correlation table* or a *table of double entry*, what we have done is to construct **two sets of frequency tables** of lengths between stated limits of breadth, and of breadths of leaves whose lengths lie between certain boundary values. Then for each interval of breadth we can find a typical length, commonly the mean: do this, and you will often find that these mean lengths lie more or less closely on a straight line LL': this line (fig. 49) shows how on the whole the lengths change with respect to the breadth of leaves—it is the *line of regression of the length on the breadth* of this kind of leaf. Similarly, making length the standard of reference—the *subject*, it is sometimes called—we can find typical values of the breadth—the *relative*, and get a *line of regression of breadth on length* BB'.

If these lines are compared with lines showing average length and average breadth (close through the intersection M of which they will pass), the original meaning of the word **regression** can be seen. Leaves broader than the average are seldom long in proportion; they have a tendency to "step back" to the average as regards length, and so the line of regression of length on breadth tends away from the line of equal variation towards the line of average length. Similarly with very long leaves; they are seldom broad in proportion, and it is significant that we never think of a long leaf broad out of proportion to its length—we naturally regard it as a rather stumpy, very broad leaf! Thus the line showing the general increases of breadth as length increases tends towards the line of average breadth. And so each of these measures tends to regress to its respective "normal" value as the other departs from its average. We seem to see here two general tendencies at work—the tendency to similarity of form, the balance of dimensions which best suits the individual; and the tendency to get

back to the dimensions which have proved "natural" for the group as a whole.

Ex. Distinguish between the procedure described above, and that used in finding typical ratios in **6.22**.

7.622. The typical distances of the actual values from either of these regression lines might, in a general way, be taken as an inverse measure of correlation between length and breadth of the leaves: but this is a loose use of the word; what is thus measured is merely the consistency, or average closeness to type, though not to the type represented by the line through the origin and M, the mean of both measures—the type in which the increase of one variable is proportionate to the increase in the other. (Cf. the deviations of **2.311**, and the average error of **6.4**.) It would clearly be more satisfactory to get some relation between these lines themselves. Smallness of the angle between them would indicate greater closeness of interrelation, i.e., a tendency for similarity of individual form to predominate. A convenient measure of this **difference of angle** (for it fits in with other quantities that are of significance) is the square root of the ratio[1] of the smaller slope m_1 to the larger m_2. If these are equal, the value of this square root is 1, the number which indicates perfect correlation (**6.41**). If there were no increase of length with breadth, LL' would be parallel to the x axis, *i.e.*, $m_1 = 0$, and there would be complete indifference of length to changes in breadth; similarly if the leaves were on the average equally broad whatever the length, $1/m_2 = 0$. If we could imagine leaves for which any increase of breadth coincided with a decrease of length, and conversely (say, leaves in which a tendency to constant area of surface predominated, irrespective of form), the lines of regression would both slope to the left; and the correlation would be regarded as negative. (These different types of cases can frequently be distinguished merely from the arrangement and relative size of the numbers in the correlation table, which thus gives direct a rough idea of the relation of the variables which is worth having.)

1 The actual values of m_1 and m_2 depend on the scales chosen for the variables: that of their ratio depends only on the frequencies.

7.623. We can also take the next step where a third measure, say, the thickness of the leaf[1] is considered. We build on the method for two variables : for each thickness interval we can find a coefficient of correlation between length and breadth ; for each length, one between breadth and thickness; and so also for each breadth, one between thickness and length. But how can we **combine** these **three coefficients** into one ? This can be done only by launching out into the sea of statistical method proper. An example of four-fold correlation, such as would be required if we added weight of leaves to the variables in the above example, is given by Brown and Thomson, *loc. cit.*, p. 146: a glance at the summary of the work given there will show how complex the problem has become.

1 How can you measure the thickness of leaves ? In the physics laboratory you should be able to devise a way of doing this. If you fail, measure something easier, e.g., the stalks of the leaves : or take an easier set of measures.

CHAPTER VIII

PROBABILITY

Let us now apply this method of representation of distributions of events by frequency curves to cases where the frequency with which events occur can be calculated. Our purpose is to find formulæ which may fit such distributions (**8.3** Exs. 2, 3). We have to make ourselves familiar with **a new typical curve,** the binomial curve, and so be able to judge if, by modification of this curve, or by combination with others, the frequency-distribution observed and the curve got from the formula fit one another in some degree. Cf. **2.31, 6.4.**

8.1. PREDICTION OF EVENTS: The simplest of these problems are those resulting from the spinning of coins; but as the probabilities (p. 3, VI v) of getting head or tail in any one spin are each $\frac{1}{2}$, there will be less tendency to confusion if we consider the spinning of symmetrical tops with three faces, these faces being coloured, say, red, white, green. Let us denote the event of the top falling on these faces by *R*, *W*, *G*, respectively. Then the probability of any one of these events occurring when one top is spun is $\frac{1}{3}$, *i.e.*, $\frac{1}{1+2}$ according to the definition. This is a *simple event.* If we spin a number of such tops simultaneously (or if we spin a top several times in succession, which is numerically equivalent to the former act), any one of the ways in which the tops fall on their faces constitutes a **compound event,** and it is interesting to predict how often any such event will happen in the way that is specified, *e.g.*, if we spin two such tops, we may expect[1] to get them both falling on the red

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1 This expectation is of course only realised in the long run after a very great number of pairs of throws. (Cf. **6.13.**)

faces once out of every nine times: for the chances[1] of each top giving R are $\frac{1}{3}$, and therefore of both giving R are $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

8.11. This is the language of everyday. The facts may be seen more clearly by substituting for vague "probability" the idea of relative frequency. (*Absolute frequency* is just the number of times an event actually occurs—what we have already spoken of merely as "frequency".) People think of the probability of any event occurring as just the ratio of the number of times it will probably happen to the total number of occasions on which it may happen. The idea becomes clearer and is related to the precise definition of p. 1, VI v, if we speak of the **relative frequency** of the event. Thus if one of the tops is spun 300 times, R will probably occur 100 times. When we are dealing with observed occurrences the natural definition is *relative frequency*

$$= \frac{\text{number of actual occurrences of the event}}{\text{number of possible occurrences of the event}}.$$

Thus, to return to the example of two three-faced tops spun together, let us consider the occurrence of $2R$ in a convenient number, say 9 pairs of spins. In these spins one top gives R three times, and for the occurrence of the second R we need consider only these three spins. In these the likelihood is that only one spin will be R; and so out of the nine possible occasions of getting two Rs it is likely that the event will occur on only one, *i.e.*, the relative frequency is $\frac{1}{9}$. The reasoning can easily be repeated for more complex events.

Ex. 1. Calculate the probability (or relative frequency) of getting two heads (or indeed any specified event, for all four are equally likely) when two coins are tossed.

Ex. 2. If two five-faced tops are spun, each having three red, one white, and one green face, find the probabilities of getting (i) two red, (ii) two white, and (iii) one red and one green, at one pair of spins.

Ex. 3. A tape is held so that the probability of an athlete clearing it is 2/5 What are the probabilities of his clearing it three and four times in succession through his successes coming at the end and at the beginning of two succeeding sets of five jumps? Can you interpret this as a test of the reality of improvement in his form if he makes several successful jumps?

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1 Is it correct to say "chance" or "chances"? If both are correct, which is the more suggestive?

8.21. ALL EVENTS: THEIR REPRESENTATION: Our next step is to represent graphically these facts about probability, in order to get an easily appreciated comprehensive view of all the possibilities in events of a given type. Take as an example the spinning of four of the three-faced tops. As to *W* (a white face falling on the table) five events are possible, *viz.*, it may occur for 0, 1, 2, 3, 4 tops. Consider the probability, *i.e.*, the relative frequency of each of these events.

For the simple event, no *W*, the relative frequency for each top is $\frac{2}{3}$; and therefore for the compound event, no *W*, (*i.e.*, no *W* for each of the four tops) the relative frequency is $(\frac{2}{3})^4$, *i.e.*, $\frac{16}{81}$.

The event, one *W*, may happen for each of the four tops[1], therefore we consider its relative frequency for any one top, say, the first, and multiply that by four. For the first top the relative frequency of *W* is $\frac{1}{3}$; then no *W* must occur for the remaining three tops and the relative frequency of this is $(\frac{2}{3})^3$; thus the relative frequency of the compound event, one W, is $4 \times \frac{1}{3}(\frac{2}{3})^3$, *i.e.*, $\frac{32}{81}$.

Similarly the event $2W$ may happen in ${}_4C_2$ ways, and its relative frequency is easily seen to be $6 \times (\frac{1}{3})^2 \times (\frac{2}{3})^2 = \frac{24}{81}$. So for $3W$ and $4W$ the relative frequencies are $4 \times (\frac{1}{3})^3 (\frac{2}{3})$ and $(\frac{1}{3})^4$, *i.e.*, $\frac{8}{81}$ and $\frac{1}{81}$ respectively.

8.211. Commonsense tells us that in considering the above five events with regard to the tops falling on their faces, we have considered all possible cases of any kind: this supplies a **test** for the correctness of our arithmetical results; for from the definition of relative frequency, the sum of the relative frequencies of the events specified in

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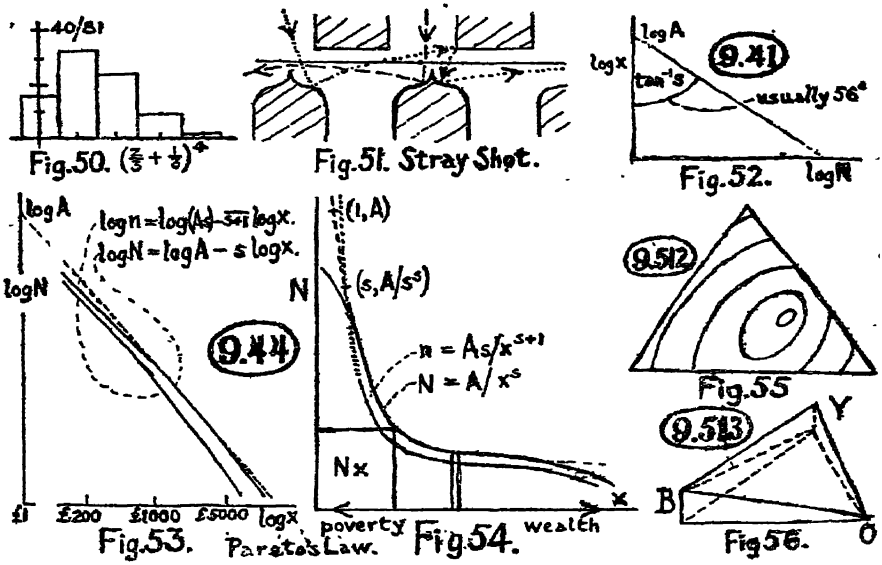
1. In considering one W, its occurrence for *any one* top is **exclusive** of its occurrence for any other (otherwise the result of the spin would be 2W, or 3W, or 4W), and the relative frequencies of exclusive events having been found separately must be *added*. [The multiplication of the probabilities of the **independent** parts of a compound event as explained in **8.1** is quite another thing (cf. Smith's "Algebra" p. 514): the "other independent event" of p. 3, VI (v) becomes just a part of a compound event.] The mutually exclusive events are different ways in which the event described as one W can occur, *i.e.*, the *description* is inclusive, and so the several events can be taken as one.

any way (by the green or by the red, say) which gives all the events without any repetition, must be unity. In this case we see that it is easiest to regard $\frac{1}{81}$ as the "unit" of relative frequency, and then 16+32+24+8+1 give 81.

8.22. These results can be represented as a column or other type of graph. As the series of events is discontinuous, it seems more reasonable to represent the values simply by points, *i.e.*, ordinates, But the **column graph** is used, as the important thing to consider is what happens when there



is a large variety, and therefore an almost continuous gradation, of possibilities; also the above arithmetical check finds an important geometrical interpretation in that the areas of the columns sum to unit area. This fact of the area under the curve being unity is of special importance when that curve is generalised as above for a great number of practically continuous possibilities. (Cf. the table referred to in **7.55**). The actual construction of the column-graph (fig. 50) presents no difficulties.

Ex. Four tops with two faces white and one green are spun together repeatedly. Work out the relative frequencies with which all possible events occur (i) for the white and (ii) for the green faces. Draw graphs to represent these frequencies.

8.3. THE BINOMIAL EXPANSION: The student who is critical will have noted that the successive relative frequencies in the example worked out above are $(\frac{2}{3})^4$, $4(\frac{2}{3})^3\frac{1}{3}$, $6(\frac{2}{3})^2(\frac{1}{3})^2$, $4(\frac{2}{3})(\frac{1}{3})^3$, and $(\frac{1}{3})^4$, and that these are just the successive terms of the expansion of $(\frac{2}{3}+\frac{1}{3})^4$. This is in agreement with the last remark that the relative frequencies total unity, and it suggests a generalisation that can be readily tested and accepted, *viz., if a simple event may occur on* n *occasions according to a definite relative frequency* q, *then the relative frequencies with which the event occurs 0, 1, 2, 3,.....*n *times are given respectively by the terms of the expansion of the binomial* $(p+q)^n$, *where* $p+q=1$. (Cf. **1.7**).

Ex. 1 Restate this generalisation substituting the definite terms of any of the above examples for the general expressions used here.

Ex. 2 Try to find a binomial expression that will fit the frequency distribution of diphtheria according to age of patient, **7.31** Ex. 5. (This and similar fitting you may attempt after experience with the exercises suggested in the next paragraph.)

Ex. 3 Try to find two binomial expressions the sum of which give values closely corresponding with the frequency distribution of clever students given in **7.32** Ex. 2. (cf. **7.54**). Find the average error of the values given by your *formula* (*i.e.* the *observed* frequencies are here the standard of reference). Cf. **6.4**.

8.4. THE BINOMIAL MACHINE: The reality and the significance of this distribution of the frequency of actual events according to the binomial "law" may be seen somewhat vividly by considering the working of the device represented in figure 57. This is called the probability machine, or more suggestively, the binomial machine.[1] The general idea is that a great number of small objects is passed through the machine, and that repeatedly the "choice" is given them of going to one side or the other according to a definite preference.

To effect this "definite preference" there are a number of moveable horizontal bars (thirteen in this machine) of

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1 The design for this machine is given by Karl Pearson in the Phil. Trans. R. S: A. 1895 Plate 7. The machine shown in figure 57 is about 4 feet high. Note the device for emptying the machine: it works very well with shot, but seed is liable to stick. All the measurements given in the text are adjusted for a space 0·4 cm. deep below the frame at the foot of the columns.

rectangular section, pressing close up to a sheet of glass: down and across these bars are cut slots of equal breadth, say 1″, to half the thickness of the bars. The spaces between these slots are tapered upwards to an edge in some suitable way so that these spaces may be used to divide, in the definite ratio selected, the objects falling through the slots of the bars above. Thus the objects passing through the single slot in the topmost bar are divided in the chosen ratio between the two slots in the next bar, and then between the three slots in the bar below; and so on, until they are collected finally in the fixed slots at the bottom of the machine. If the apexes of the spaces between the moveable slots are adjusted so that they divide all the slots in one ratio, say, 3 : 7, as in the illustration, then the objects passing through each slot are so divided ; and this happens at each bar. This means that the objects are separated twelve times in succession in this ratio, and so they are finally collected in quantities which are proportional to the thirteen terms of the expansion of $(\cdot 3 + \cdot 7)^{12}$.

This can be worked out in detail thus: taking the quantity passing through the first slot as 100, 30 go to the left and 70 to the right; of the former 9 go to the left, 21 to the middle slot of the third bar; into this slot go also ·3 of the 70, i. e., 21, while 49 go into the righthand slot of the third bar. And so on.

The process becomes clearer. still if it is generalised, the quantity passing through the top slot being taken as unity, the ratio of division as $p : q$; then the quantities passing through the slots of successive bars are

$$\begin{array}{ccccc} & & p & , & q \\ p.p & , & q.p+p.q & , & q.q \ ; \quad \text{i.e.,} \quad p^2 \ , \ 2pq \ , \ q^2 \\ p.p^2 \ , & q.p^2+p.2pq, & q.2pq+p.q^2, & q.q^2 \ ; \ \text{i.e.,} \ p^3, & 3p^2q, \ 3pq^2 \ , \ q^3 \\ & \text{etc} & & & \text{etc.} \end{array}$$

8.41. THE ERRORS OF THE MACHINE: It is well worth while considering the working of this machine in some detail, for the results got from it do not closely accord with theory; an instrument of this sort would not be allowed in any physics laboratory. The reasons for this discrepancy

bring out well the nature of some of the problems that are considered in statistics; here we have "writ large and plain" the defects which occur in all measuring instruments, and which have to be considered when the finest of these instruments are used to give results up to the limit of their accuracy; and, of course, in most sets of observed quantities of any kind we have to allow for disturbing factors in trying to determine the chief factors that affect the measurements. Cf. also **1.431** Note.

The ogival shape of the dividing blocks between the slots in this particular machine was adopted because it was realised that it was of great importance to get the seed or shot passing through the machine **spread** as **uniformly** as possible across the slots in downward succession; otherwise the division of the seed in the slots below would not be at all according to the ratio set in the machine. It was clear too that the best shape of dividing block would not be the same at the bottom as at the top of the machine, because of the difference of speeds of any object earlier and later in its fall; also the best shape for seed would not be the best for shot, because of the difference of density. Amid all these conflicting considerations the shape adopted could be only a guess as to what might be most effective. A careful inspection of the illustration,[1] especially of the moveable bar marked 6, will reveal another less important precaution; the dividing edge is placed $\frac{1}{10}''$ below the top of its bar, and therefore well clear of the sides of the slot above: this was merely to make it possible for fairly large seed to pass through the smaller opening when the ratio set was 1:9, or even 2:8. The actual effect of all this, however, was that, when shot were passed freely through the machine, the force of impact on the shoulder of a block sometimes caused a pellet to rebound over the next block into a slot which it ought to have been impossible for it to enter! (Fig. 51).

It appears that the result given by such a machine will always tend to deviate from theory in that **the central**

1 It is worth while using a lens to inspect accurate figures and to read the graduations of good scales.

columns will be **shorter** than they ought to be. This is due to the fact that the shot starts from a central position and there is a general movement towards the sides: thus in the medley of interference between the pellets rebounding from the sides of the slots the impacts directed outwards will preponderate, with the result that has been stated.

This was so especially when the funnel at the top was filled with shot and suddenly opened. The resulting downpour, like any other, was interesting, but depressing; for it interfered much with liberty: there was little of "choice" or "preference." Shot poured gently through gave better results, which varied according to the speed with which it was poured.

The device finally adopted was to tilt the machine backwards till it was **almost horizontal** and the shot was then passed through gently; but even so there were difficulties in securing uniformity of distribution, *e.g.*, care had to be taken that the shot did not get blocked owing to a bar bulging out slightly beyond the bar above it.

As a consequence of this need for avoiding the effect of impacts, the funnel at the top turns out to be superfluous though it is not inconvenient. In any case there is a real difficulty in getting the dividing edges placed accurately enough to give a really consistent division of the objects passing through every slot.

Ex. 1. The results of passing shot through the machine when 12 bars were set to the ratio 8 : 2 are shown in the following table. An easy way of **calculating the theoretical values** of the heights of the columns is also indicated. Notice how here, as in all these experiments, no matter how the shot is poured through, the tendency to excess accumulation at the sides in this machine is clearly marked.

In the expansion of $(\cdot 8+\cdot 2)^{12}$ each term is got from its predecessor by multiplication by the extra factors in the coefficient ${}_{12}C_r$, *viz.*, $\frac{12}{1}$, $\frac{11}{2}$, $\frac{10}{3}$, $\frac{9}{4}$,.....and also by $\frac{\cdot 2}{\cdot 8}$, *i. e.*, $\frac{1}{4}$. These multipliers are shown in the lefthand column, and the decimal value is noted, if need be, to the left of this; the meaning of the rest of the table should be evident.

				Theory Per cent. (from preceding column)	Measurement Actual.	Per cent.	Excess.
$\log (\cdot 8)^{12} = \bar{2}\cdot 8372 \cdot = \log \cdot 0687$							
	$\frac{12}{4}$	$\cdot 4771$					
		$\bar{1}\cdot 3143$	$\cdot 2062$				
	$\frac{11}{2.4}$	$\cdot 1383$			[·2 Stray shot]		·3
		$\bar{1}\cdot 4526$	$\cdot 2834$	6·9	16·8	25·2	18·3
				20·6	17·4	26	5·4
	$\frac{10}{3.4}$	$\bar{1}\cdot 9208$		28·3	11·9	17·8	—10·5
		$\bar{1}\cdot 3734$	$\cdot 2362$	23·6	7·9	11·8	—11·8
				13·3	6·2	9·3	—4
$\cdot 5625 =$	$\frac{9}{4\,4}$	$\bar{1}\cdot 7501$		5·3	3·0	4·5	— ·8
		$\bar{1}\cdot 1235$	$\cdot 1329$	1·5	2·1	3·1	1·6
				·3	1·0	1·5	1·2
etc.			etc.	·1	·3	·5	·4
				0	·1	·2	·2
				0	0	Average error	
				0	0	(without	
				0	0	regard to sign)	
						5 per cent.	

Note as a check how the logarithms of the factors $\frac{12}{4}$, $\frac{11}{2.4}$, ... change regularly. The successive differences are ·34, ·21 ·17, ·15, ·14, ·13, ·14 ·15,...(cf. **2,18**)

Ex. 2. Repeat the above investigation for the following measurements obtained with a 3 : 7 setting of the machine ; draw the graph of the theoretical values, and compare it with the curve given by the machine : 0, 0·1, 0·5, 1·8 3·9, 5·3, 8·3, 9·7, 10·4, 10·2, 5·7, 3·6, 0·3, [9 balls].

Ex, 3. Examine for the same setting, 2 : 8, of 12 bars. the following measurements of the distribution of shot poured through the binomial machine by different persons :

T	0, 0, 0, ·2, ·5, 1·4, 2·3, 4·6, 7·5,	10·4, 15·6, 18·7	[·3, 2 balls.]
B	0, 0, 0, ·2, ·5, 1·7, 3·0, 5·8, 9·8,	12·2, 16·3, 11·2	[·5, 13 balls.]
T'	0, 0, 0, ·2, ·5, 1·5, 2·6, 5·4, 7·9	11·3, 16·4, 15·6	[·3, 6 balls.]
B'	0, 0, 0. 2, ·6, 2·0, 3·5, 6·5, 10·4,	13·0, 16·7, 7·7	[·6, 8 balls.]
G	0, 0, ·1, ·3, ·8, 1·5, 3·3. 6·2, 10·1,	12·6, 16·8, 9·4	[·6, 8 balls.]
D	0, 0, 0 ·2, ·4, 1·4, 2·4, 4·3, 6·9,	9·9, 15·2, 20·9	[·3, 0 balls]

(D was noticeably hasty in pouring the balls through.)

8.411. Another device used sought uniformity of distribution of pellets in the slots by making these longer, and so giving the pellets more opportunity to spread. The **bars** were set **in pairs** with slots directly one above the other so that a division of the shot took place only at alternate bars this was an improvement, but it reduced by half the number of columns of the machine used.

Ex. Check the values of the average error of the heights of the columns of shot in the following measurements, the bars being set in pairs to represent $(\cdot 3 + \cdot 7)^6$, and the shot being poured through the machine as indicated.

		Av. error.
Free rush	[9 balls, 0], ·8, 1·3, 8·1, 4·5, 8·7, 7·8, 33·9, [·4, 8 balls.]	16 per cent.
Partial rush	[3 balls,], ·3, ·8, 3·4, 7·4. 12·6, 14·4, 23·4, [·2, 10 balls.]	8 per cent.
Poured gently	[1 ball] ·1, 1·0, 3·4, 9·9, 17·5, 17·4, 10·8, [1·1, ·1, 1 ball.]	2 per cent.
„ „	[1 ball]. ·1, ·9, 3·4, 9·6, 16·0. 17·4, 12·9, [·6, ·1, 1 ball]	3 per cent.
With seed corresponding results were	·2, 2, 5, 11, 16·5, 21, 24·5	6 per cent.

[Note that the results here and in **8.41** Ex. 3 could not fittingly be combined to give an average result for the machine. Compare this fact with Dr. Price-Jones' attitude mentioned in **7.53** against adding the frequencies in the same class for different cases of anæmia.]

8.412. One defect of the machine, constructed as here described, is that the entrance for the shot is fixed rigidly. The columns at the bottom of the machine must be fixed, and it is usually impossible to adjust the intermediate bars so that the uppermost moveable bar comes into the proper position with respect to the entrance. This results in one of the dividing bars becoming ineffective.

(Note the lack of symmetry in the figure on p. 298 of Yule's "Theory of Statistics": also, though less noticeable, in that referred to in **8.4**, f. n.)

8.42. An Improved Machine: Possibly the best shape for the dividing blocks is simply triangular, especially lower down in the machine where the risk of jumping from the shoulders of the blocks is greatest; and the lower corners of the dividing blocks should be cut away: it would probably leave too great freedom to the pellets were the blocks removed save for the top surfaces. But all this can be determined only by experiment. Probably the most consistent results would be obtained, however, only if the machine were re-designed so as to allow water to flow gently through it: the bars, when they have been adjusted could be clamped together to prevent leakage—were they arranged so as to give a low cascade, some measure of the picturesque might be introduced into the machine at the same time as it becomes more satisfactory in giving results which accord more with theory. By this device also difficulties due to lack of fit between the moveable bars and the glass front would be removed; for the latter would become unnecessary.

8.5. The binomial machine may be compared with **Galton's Quincunx**, which is thus described in Whittaker and Robinson's "Calculus of Observations", p. 168: "Into a board inclined to the horizontal about a thousand pins are driven disposed in the fashion known to fruit growers as the *quincunx*, i.e., so that every pin forms equilateral triangles with its nearest neighbours. At the top of the board is a funnel into which small shot is poured. The shot in descending strikes the pins in the successive rows, each piece being deviated to right or to left at every encounter with a pin." It is suggested that the results of the working of the quincunx are satisfactory; if so, it seems that the error due to sideways displacement is compensated by the lack of definite separation: but the experimental record to which reference is made is not available in India, and it has not been possible to examine the matter here. However, the two machines are really **different in principle**: in the quincunx the effect is produced by random impacts—it does not seem essential that the pins should be arranged regularly; this is but a way (as with the fruit trees) of guarding against undue crowding and sparseness of obstacles: in the binomial machine the idea is to control the shot in a definite way. The quincunx gives only the symmetrical binomial curve. This can also be obtained by any device which allows particles, *e.g.*, flaky seed falling from an opening, to spread at random before being collected in a row of compartments set side by side.

Fig. 57. The Binomial Machine, set to $(\cdot 3 + \cdot 7)^{12}$.

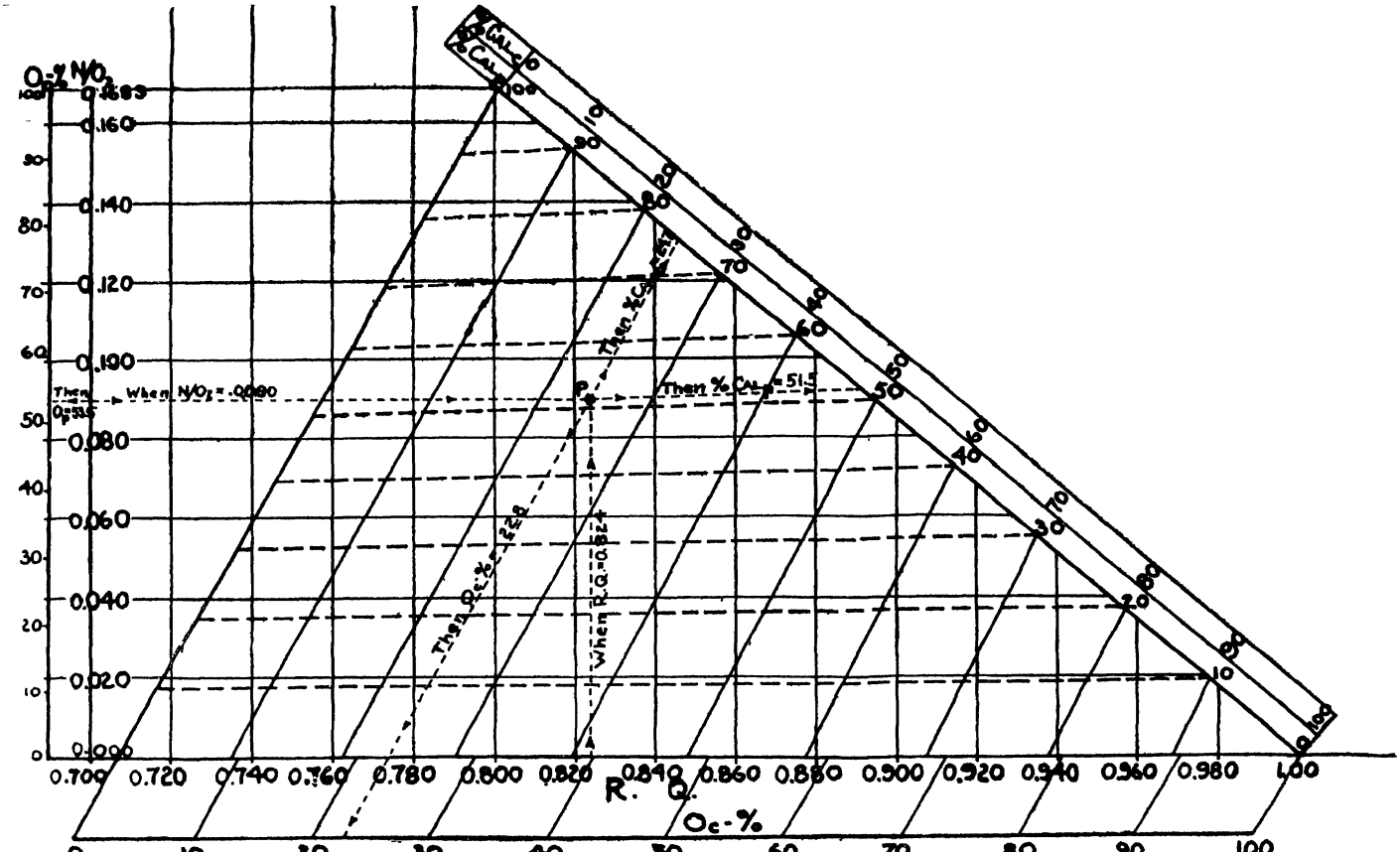

Fig. 58. Carbohydrate and protein : oxygen used to oxidise, and proportion of heat from, each of these. (Michaelis).

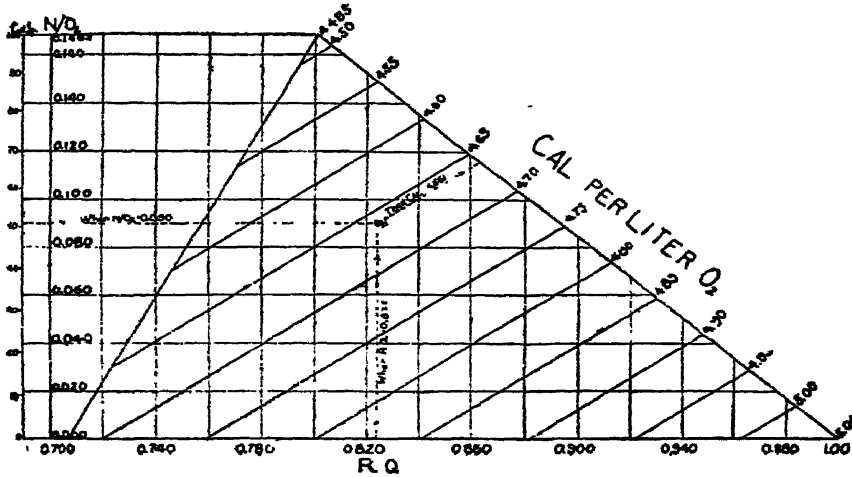

Fig. 59. Heat produced in the oxidising of carbohydrate, protein, fat in any proportions. (Michælis) 9.52

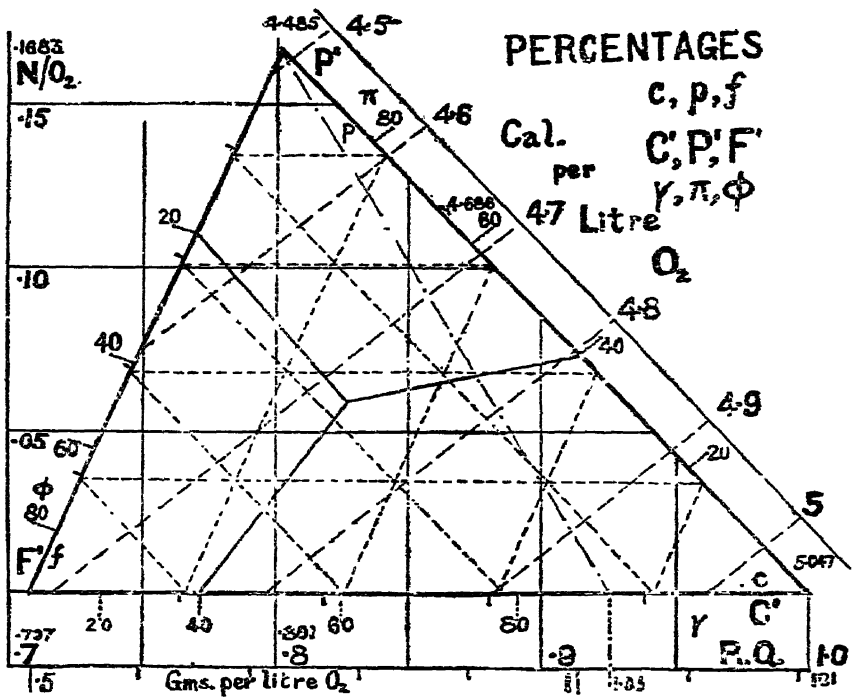

Fig. 60. Generalised Diagram showing Quantitative Relations in Metabolism.

(Note that each side of the triangle carries three scales, the two on the outside being distinguished by graduations, whole and broken, and also of different lengths.)

CHAPTER IX

SPECIAL GRAPHS

9.1. LEPROSY CURVES: **Quantity** is frequently made **the basis of classification** even when we are not thinking of frequency curves: we speak of octogenarians and of millionaires, or we reject men below a certain height for the

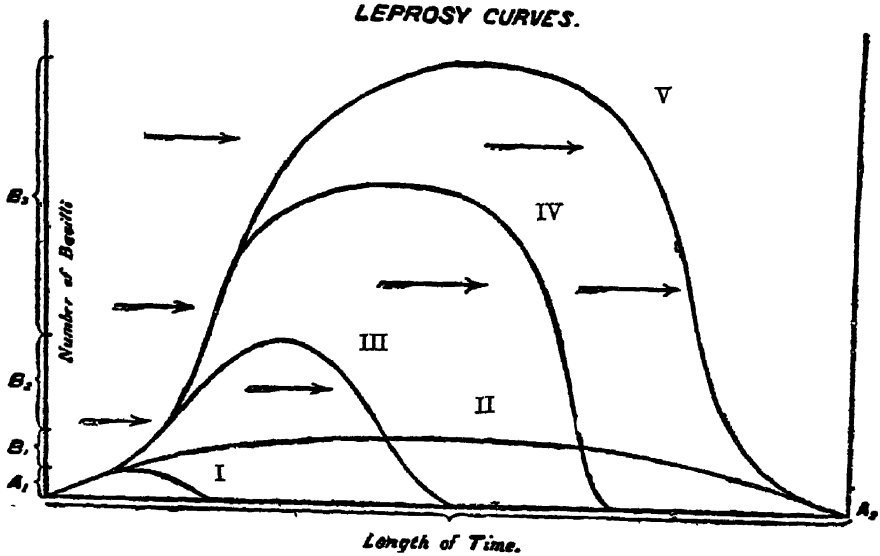

Fig. 61. General Characteristics of the Course of Leprosy in Typical Cases.

"A =Nerve (anæsthetic) leprosy.
A_1 =Primary nerve leprosy.
A_2 =Secondary nerve leprosy.
→ =Reaction-producing causes.
B =Skin (nodular) leprosy.
B_1 =First stage of ditto—few bacilli found in the skin.
B_2 =Second stage of ditto—more bacilli found in the skin.
B_3 =Third stage of ditto— very marked, generalised lepromatous infiltration, abundant bacilli."

police force or the army. An interesting and important example of the use of a graph to aid in clear arranging of this sort is the modern classification of patients suffering from leprosy. (Figure 61.[1])

1 From a pamphlet on leprosy by Dr. Muir of the Calcutta School of Tropical Medicine and Hygiene: also "Lancet" **208** 171, 1925.

For convenience the curves shown in the figure have been numbered from I to V; otherwise the diagram has been reproduced as originally given. **No scales** are shown, for the graphs are meant to be merely typical. This vagueness is easily understood with regard to the horizontal axis, for obviously the duration of the disease will vary with the age at which the attack comes, the natural resisting power of the patient, etc. As regards the vertical axis, along which the numbers of bacilli found in a patient from time to time are represented, it may be said that the limits of the *B* types are determined by the maximum values on the curves I, II, III, and that the numbers of bacilli which may be found in the successive *B* types are respectively 2, 5, and 15 (apparently the maximum ever found) times that found in the *A* type. Then, if we can speak so definitely, why is no scale inserted? In this case the reason may be that the *lepra* bacilli are found in clumps as well as singly, and an indication of quantity must be the result of an impression rather than of actual counting. Also the absence of a scale emphasises that the diagram gives typical, and not precise, representations. Comparison with the classification given in the Encyclopædia Britannica (Ex. 2 below) suggests that we have here only the initial stages in the determination of a classification which will put the treatment of leprosy patients on a basis of much greater certainty than at present.

However, our task is to consider the general significance of the diagram, and that is clear. There are **two types** of leprosy, denoted by *A* and *B* (the latter divided into three sub-types), in which the nerves and the skin respectively are affected. Cases of the former type are distinguished only in time as primary and secondary, *i.e.* according as the resisting powers of the body are yielding to or overcoming the attack of the disease. Cases of skin (or nodular) leprosy are distinguished by quantity as described above, but it is important to notice that they have passed more or less quickly (though this is not represented on the diagram) through the anæsthetic type; hence the great importance of diagnosing early the patches of insensitive skin which characterise this form of the disease.

The general health of the individual has a marked effect on the likelihood of his incurring, and on the course of, the disease. The occurrence of fever or some other abnormal condition is represented by arrows, feathered according to the severity of this **reaction-producing cause.** This very general term is used, because the effects of the same cause are quite different at different stages of the disease: what may *predispose* to an increase of the disease in the early stages will in the latter stages lead to a marked decrease of its virulence; these *elimination phenomena* are not yet understood. But the importance of quite different treatment when the disease is gaining ground and when it is losing is obvious.[1]

For this reason there would be an advantage if the **notation** used distinguised between *type* and *stage*, or, as these words overlap, *number of bacilli* and *time*. This can be effected very simply by retaining the notation for the four types A, B_1, B_2, B_3 at the primary stage, and adding a dash to each letter at the secondary stage: thus there would be three "stages" of decreasing virulence B_3' (read "B three dash"), B_2', B_1', A'. There seems to be no advantage in drawing the arrows parallel to the time axis; this suggests a uniformity which does not exist.

In this diagram leprosy is represented as a **self-healing disease**: cure is effected by forces that are not understood. This representation makes clear how contradictory results from the same treatment in apparently similar cases may arise. Also the course of the disease has the elusiveness that follows from the freedom to move in a plane, not the simplicity of progress along a straight line. (Apparently

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1 The following quotation from the Lancet is of interest: "The object of the removal of predisposing causes and of special treatment is to flatten the curve, to bring down the crisis as soon as possible, and to hasten the downgrade of the curve. Most of the deformities of lepers are due to *lingering on the down-grade.* To avoid this, vigorous treatment must be applied when once the signs of the disease have begun to diminish."

there can be only one maximum on the curve.[1]) Curve IV represents one of the possible intermediate cases. No fatal cases are represented, though death might result from a reaction-producing cause or otherwise: the curve representing this would end without reaching the time axis again.

Ex. 1. "Many cases follow a course represented by a flat curve for a certain time (it may be many years) and then, due to sudden lowering of the resistance, the number of bacilli increases rapidly and the curve passes upwards abruptly into the *B* area." (Dr. Muir) Represent this type of case on the diagram.

Ex. 2. In the Encyc. Brit. **16** 479d the types of leprosy distinguished are (1) nodular, (2) smooth or anæsthetic, (3) mixed. No relationship between these appears to have been clearly seen. Interpret this classification by the diagram.

Ex. 3. Have the intersections of curves III and IV with II any significance?

This is a question a mathematician should ask. The answer might be in the negative, for the intersecting curves represent different cases. But the intersections represent this at least, that the resisting power of individuals equally afflicted may be very different; and possibly, that cure of a severe case is often easier than that of a mild one. More searching questions are: What significance has the slope of the leprosy curve?[2] Why does curve V coalesce with II, instead of cutting it as IV does?

9.2. A BLOOD CHART: The following physiological example will show how one diagram (fig. 62) may be used to represent the relations between as many as six variables which are related among themselves in some

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1 This is so: one of the objects of the illustration is to show that very rarely are there large fluctuations in the course of the disease, though phases of quiescence, reaction (due to breaking down of lepramatous tissue), and resolution may be superadded again and again: cf. the *general trend* and the *deviations* of **2.311**. Yet, if there *were* distinct successive maxima in the course of a disease, it would be easy to **modify the notation** to show the number of times a maximum is known to have been passed, e.g. by prefixing a numeral to the symbol. Thus $3B_2$ would mean that the disease had passed through two known maxima and was at the primary stage of a third fluctuation and at the level of intensity of the type B_2; $3B_2'$ ("three *B* two dash") that it was at the stage of the same fluctuation in which the disease was abating, but of the same typical intensity.

2 In the Journal Biol. Chem. **59** 426 the slope at a point in a blood-system diagram is interpreted as showing direction of diffusion, and thus a representation of the respiratory cycle is got (this is transferred to all the 105 charts referred to in **9.21**). In the same diagram, the axes being marked with scales for Total O_2 and Total CO_2, the slope of a straight line gives the *R.Q.* (**9.521** ii).

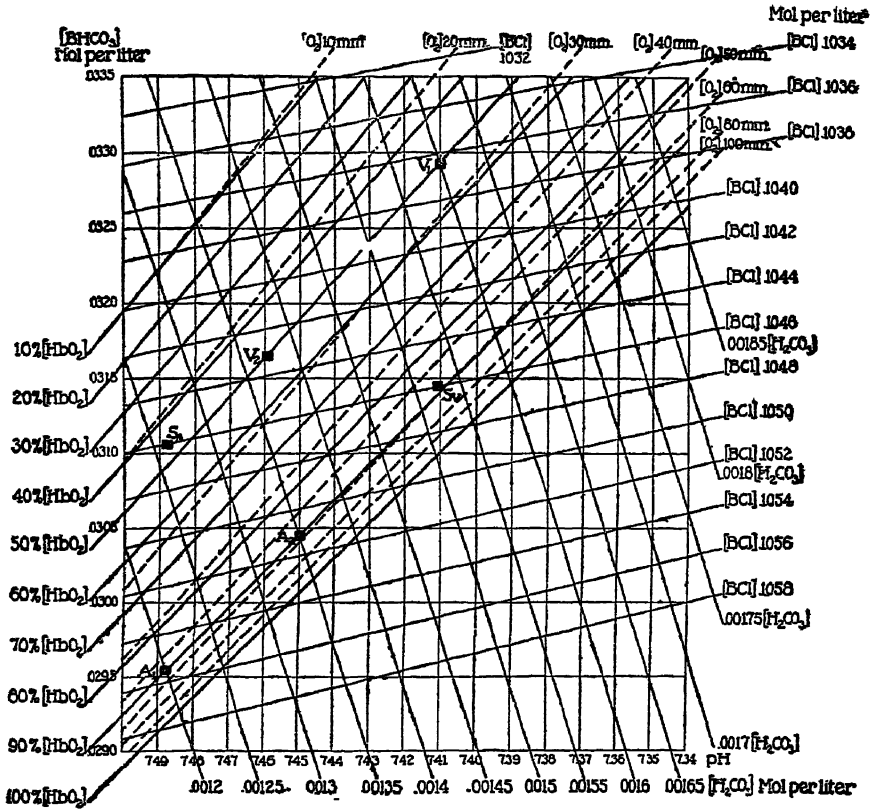

Fig. 62. Blood as a Physicochemical System.

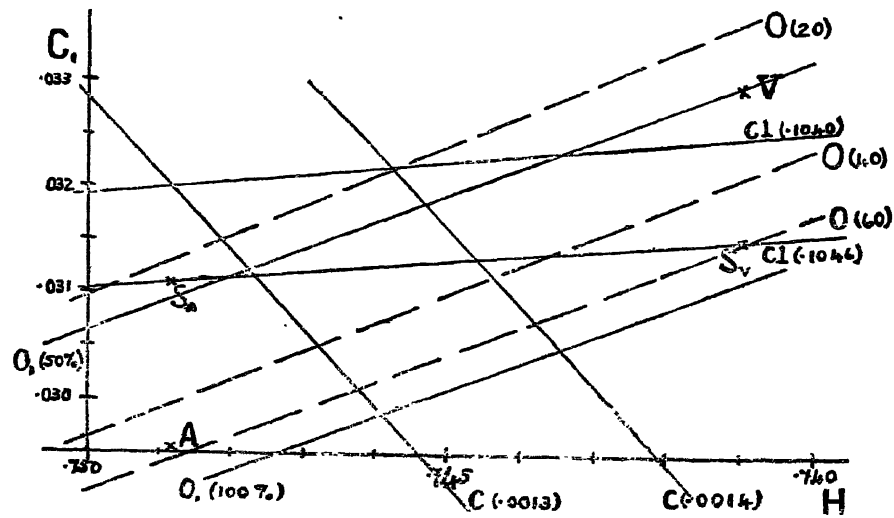

Fig. 63. Simplification of part of Fig. 62.

way that is not yet fully understood.[1] In studying the composition of the blood under different conditions physiologists have noted six substances and properties which can be measured: these we may denote by O, O_1, C, C_1, H, Cl.[2] From these H and C_1 are chosen for representation along the perpendicular axes of reference. It is then found that specimens of blood which contain given fixed quantities of the substance O are represented by points (giving the quantities of H and C_1 they contain), which lie along certain lines: three of these are shown by the broken lines in the simplified figure, 63. So also for specimens containing equal quantities of O_1; they are represented along lines which happen to be not very different from the O lines. The C and the Cl lines obtained similarly are easily distinguished from these, and from one another.

We cannot attempt to discuss here the meaning of this diagram,[3]—an intersection nomogram, it may be called. But the main point is clear, *viz.*, that, given the quantities of any two of these constituents, the quantities of the other four are determined; for the point which represents the amounts

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1 This phrase was written before the following sentences from the Journ. Biol. Chem. **59** 400 were read. With reference to a nomogram containing seven scales given on p. 387 therein (**9.24**), it is said, "From a logical standpoint the figure is one among several possible complete expressions of the nature of blood as a physicochemical system, in accordance with present knowledge. We believe that it contains neither more nor less than the necessary and sufficient number of scales, although, within limits, a different choice of variables is open." In the complete nomogram eighteen scales are inserted to show quantities of importance in physiology; but no new mathematical principle is involved in drawing these. Of the whole diagram Henderson says, "an alignment chart is probably the only means of presenting such a great mass of quantitative information in compact form." It is equivalent to the whole 105 charts, noted in **9.21**.

2 For the sake of the student who has dabbled in chemistry it may be mentioned that these denote respectively O_2 and HbO_2 of the whole blood, H_2CO_3, $BHCO_3$, pH and BCl of the serum; *i.e.*, oxygen, oxyhæmoglobin, free carbonic acid, combined carbonic acid, hydrogen-ion concentration, combined chlorides (with potassium, sodium, calcium).

3 The main comments of L. J. Henderson, the author of this diagram, are reproduced with the complete original diagram in Pearl's "Medical Biometry and Statistics", page 135.

of the first two substances represents also definite amounts of the other four substances.[1]

9.21. Questions arise as to why the main **axes of reference** are taken for H and C_1: if another pair were

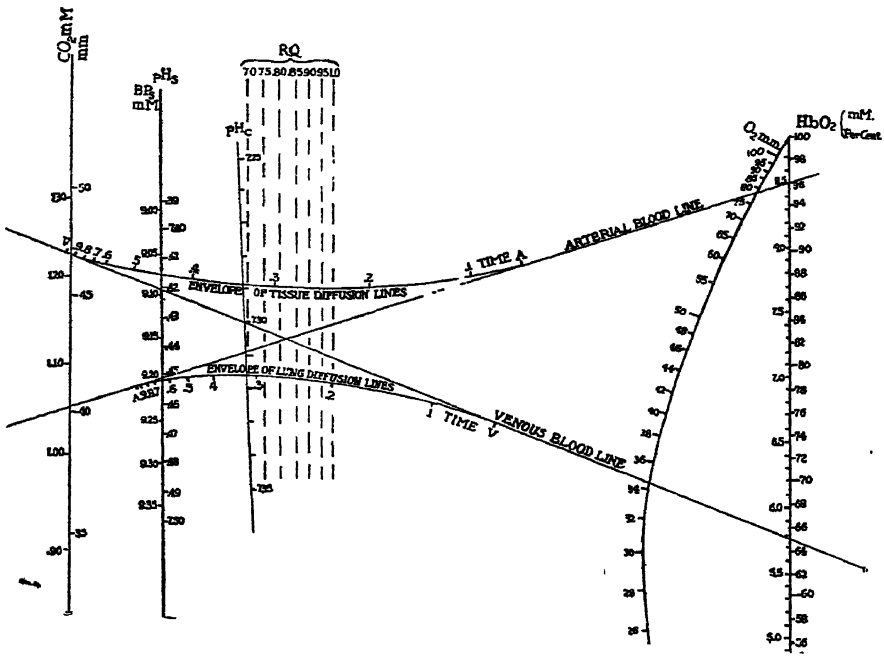

Fig. 64. Part of a nomogram representing "the law of the blood".

selected, would the diagram be simplified or otherwise? The question is a legitimate one, and might need careful investigation[2] before an answer could be given; in this case

1 This same fact is stated in the Journ, Biol. Chem. **59** 400 in terms of a line cutting the seven scales of the blood system nomogram just mentioned.

2 After writing this the investigation was found in a paper by Henderson, Bock, Field and Stoddard in the Journ. Biol. Chem. **59** 379. Part II thereof, with numerous diagrams, looked like a record of many chemical experiments. I would have passed it by, but reading a few sentences showed that the section was entirely mathematical! The 7 variables can be combined in twos in $_7C_2$, *i.e.*, 21 ways, and with each of these pairs as axes of reference can be taken each of the other 5 variables. The **105 diagrams** thus possible are **drawn from the original diagram** got from experimental data, so that students of physiology may drill themselves in looking at this complex subject from all possible points of view!

the H and the C_1 might have been selected because of, say, a complementary chemical relationship, similar to acid and base:[1] note that the C and the Cl lines in the original diagram are nearly perpendicular, though this is an accident due to the scales chosen for H and C_1 (cf. the simplified diagram); yet the C and the Cl lines are not really straight lines—the diagram is based on experimental results, not on a theory which has been discovered behind the experimental facts and by which they can be checked.

9.22. An improvement in the **clearness** of the diagram, especially in the original with its numerous horizontal and vertical lines, would result if the O and O_1 lines were marked in red; for they have reference to the whole blood, the other four to the serum only, *i.e.*, the fluid separated out when blood clots on being removed from the body. Every opportunity of representing similar properties by related directions, forms or colours should be utilised; this is but an extension of the economy of effort in attending to facts which a graphical method should effect.

9.23. One of the applications that has been made of these curves is not difficult to follow, though it can be stated here in but a crude manner. It is well known that blood from the arteries contains more oxygen and less carbonic acid than blood from the veins. Average specimens of these types of blood are indicated by *A*, *V* respectively in figure 63, the quantities of oxygen indicated with reference to the dashed lines being 56 and 26, those of carbonic acid, C_1, ·02954 and ·03293. What happens when the **venous blood loses** its **carbonic acid** after it enters the lungs?

Two parts of the blood, the corpuscles and the plasma (*i.e.*, the liquid part in which the corpuscles move) may take a share in bringing about these changes. It is known that the plasma is active only when the movement of the point which shows the change in the composition of the blood is along one of the Cl lines. In the diagram this change due to the plasma is represented by $S_V S_A$, and the corresponding

1 As a matter of fact the fundamental variables were chosen here simply for experimental convenience: acidity and carbonates are both very easily determined.

decrease of carbonic acid is from ·03144 to ·03106, only ·00038 out of the total change of ·00339, *i.e.*, one part of the elimination of the carbonic acid is due to the activity of the plasma compared with ten parts due to corpuscular action.

Ex. Is there any reason why S_A, S_V should be on the particular Cl line shown in the figure ?

9.24. The righthand part of the **nomogram** referred to in **9.2** f.n. is reproduced as figure 64 in order to illustrate remarks, in **5.5** regarding the construction of a nomogram for which a formula is not known, in **5.6** about the equivalence of intersection (*i.e.*, Cartesian) and nomographic charts, in **9.521** about a grid for the *R.Q.*; and other details mentioned elsewhere. Only four of the necessary and sufficient scales for a complete description of the blood system are given: the fundamental scale for Total CO_2 (*i. e.*, the $BHCO_3$ plus H_2CO_3 of figure 62) had to be omitted, being too far to the left; it is graduated uniformly. (The other fundamental scale in the intersection diagram from which the nomogram is constructed is Total O_2.) The horizontal reference scale in figure 62 is replaced by two scales in figure 64, one for cells and one for serum (cf. **9.22**): the latter is one of the seven primary scales got from experimental data; the former is not quite parallel to the other scales (**5.5**). Note the proximity of the HbO_2 and the O_2 scales in both figures; also that these scales have a positive slope in figure 62, and are beyond in figure 64 the parallel scales corresponding to the axial scales of 62 (**5.6**).

Ex. 1. Explain the reasonableness of the slopes of the two blood lines. (An *envelope* is a curve marked out by tangents in close succession: cf. **5.6** f.n.) More precisely, show that these straight lines correspond to ths points *A* and *V* in the Cartesian diagrams. Stretch a thread tangent to the diffusion lines at the points showing decimal fractions of the time required for diffusion, and plot on figure 63 (or 62) the points corresponding to these tangents. (Cf. Journ. Biol. Chem. **59** 415.)

Ex. 2. Taking time of diffusion for uniform horizontal reference scale, draw graphs showing the changes of each of the other variables in either diagram. (Note how the possibility of moving from tangent to tangent on the nomogram corresponds to motion from point to point in the ordinary diagram.)

Ex. 3. Transform figure 64 into an intersection diagram as nearly like figure 62 as possible.

9.25. It is interesting to note two of Henderson's cautions (hinted at in **9.21.**) with regard to his diagram, 62,

(i) The O_1 lines are more nearly straight if drawn against a background in which not only H but also C_1 is plotted logarithmically.

(ii) The C and Cl lines really have a slight curvature.

These comments lead us to consider more general types of graph paper.

9.3. SEMI-LOGARITHMIC GRAPHS: When numbers are plotted for a long series of years during which there is a large increase or decrease, an alteration which would have been important at the beginning of the series might have practically no significance at the end, or conversely. Yet if the figures are plotted on ordinary graph-paper these **fluctuations** appear equally striking to the eye at both stages; and so a false impression is given. The effect of a graph of receipts or expenditure on ordinary paper is often unduly cheering or depressing! A semi-logarithmic graph is **a more sober record** of facts.

What it is desirable to show is the change relative to the total quantity involved, and this is done if, instead of plotting the numbers themselves, their logarithms are plotted. Then a change from y_0 to y_1 appears as $\log y_1 - \log y_0$. *i.e.*, as $\log (y_1/y_0)$, which is always important, and not as $y_1 - y_0$, the significance of which varies very much. This plotting is most conveniently effected on the semi-logarithmic ruling as it appears on the right of the blackboard depicted in figure 2. The x axis is usually graduated uniformly, for time is " an ever-flowing stream"; but the y axis is on a logarithmic scale, and the values of y are placed on it in the same way as the cardinal numbers are marked on the slide rule.

9.31. To read the graphs in figure 2 in the usual way the figure must be turned so that the lefthand side is below: the time scale is then at the top of the figure. A comparison of the deathrate curves, drawn on the two kinds of ruling from the same figures in Table III (c), will show clearly the contrast between the two methods. In the space available

comparison is possible only from 1880 onwards. The apparently remarkable fluctuation from 1880 to 1883 shown on the ordinary graph is far less prominent on the semi-logarithmic graph, where it is comparable with the fluctuations about 1910 and 1920. The important fact to note, however, is that, while the ordinary graph indicates a decrease which has a tendency to fall off, the logarithmic graph shows a definite **change in the rate** of decrease at two points, about 1887, and, more definitely, at 1910; also that the rates of decrease have become distinctly larger. (The ordinary graph should be carried back to 1870, and it will be seen that for that decade it becomes almost unmanageable and unintelligible.) These facts have an obvious explanation in that "about 1888—1890 the activities of the State board of health in the study of purification and sewage treatment were at their height, and about 1910 the pasteurization of milk was adopted extensively. There have been no sudden changes in the quality of the water supplies of the State, but a steady improvement due more to protective measures than to water purification and chlorination."

9.32. On semi-logarithmic graph paper, which may be called also **ratio paper,** equal ratios[1] of change, *m*, *e.g.*, percentage changes, are shown by equal slopes; just as on uniformly ruled graph paper equal **differences** for corresponding intervals are shown by equal slopes on the same or different lines. This may be seen directly; or we can deduce it from the equation of a line drawn on such ruling. If this be $\log y = mt + c_1$, it may be written[2]

$$y = 10^{mt + c_1} = 10^{c_1} . 10^{mt} = ca^t$$

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1 Sometimes, *e.g.*, "Working Class Budgets", Chart No. 6, the value of ratios are shown by separate lines of appropriate length drawn parallel to the logarithmic scale; this is certainly provocative of thought, but it is a cumbrous device, justified only by general ignorance of such graphs. Whipple draws lines of slope 1·1 per cent. (cf. Ex. 7) on the charts showing growths of general populations; this, however, is to show the typical rate of increase with which particular instances are to be compared.

2 By comparison with $A = P(1+r)^n$ of **2.21,** note that, for respectively equal scales, $a = 10^m = 1 + r$, and so $m = \log(1+r)$. Cf. Exs. 6, 7; also **3.13** (ii).

and this is the exponential curve of **2.21**, which shows a constant relative rate of change. Conversely, if we find that a series of points plotted on this paper lie close to a straight line, we deduce that the "law" governing these points is of the exponential form: its constants can be determined for the equation $\log y = mx + c$ by measurement in the usual way, the relation of the scales for $\log y$ and x being noted; and then this equation can be transformed into any of the above forms most suitable. (Cf. **9.33** Exs. 6, 7.)

9.33. If semi-logarithmic paper is not available, a third column giving the logarithms may be added to a table of values, and these logarithms can be plotted on ordinary paper; but it is well to have **a permanent semi-logarithmic ruling** on which results may be plotted direct without the labour of looking up logarithms. This is easily obtained by ruling a slate so. It will be found convenient to make the length of the logarithmic scale corresponding to a factor 10 equal to about 4 inches. The reasons for this are two: with a larger scale it is difficult to interpolate accurately towards the lower end of the logarithmic unit, though here extra lines should be ruled, just as there are extra graduations on the slide rule[1]; and on an area the size of foolscap (or preferably more square than foolscap) it is possible to get a range of from 1 to 1000 on the logarithmic scale, and this will serve for most purposes.

Ex. 1. Calculate the amounts of two very different sums of money, *e. g.*, Rs. 10 and Rs. 100, at the ends of 20 successive years at the same rates, compound interest. Represent these values on ordinary and on semilogarithmic ruling. What conditions determine the slopes of the C. I. lines on uniform ruling?

(Note that in comparing two series of prices represented on ordinary ruling the fluctuations of each depend on the unit quantities for which the prices are taken; and so a faulty choice of units may deprive the graphs of meaning: this defect does not occur on semi-logarithmic ruling. Also for this property compare semi-logarithmic graphs with index numbers, **6.511**.)

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1 An effort should be made to make the meaning of all lines very clear, without thickening them. Whipple's graphs, referred to in **9.32** f. n. 1, would be much easier to read if the unit lines were produced slightly beyond the border, or the half unit lines broken.

Ex. 2. Compare the business of Indian cities as indicated by the amounts of cheques (lakhs of rupees) cleared annually as in the accompanying table. (Indian Year Book, 1924, p. 768.) The graphs given in this publication on pp. 838 to 848 are better drawn on semi-logarithmic ruling, but people are not yet accustomed to this form: hence the inferior representation has to be used. In his "Vital Statistics" Whipple uses this ruling freely; but, being primarily concerned with quickly improving sanitation, etc., he gives a warning (repeated in the Report quoted in Ex. 4) against using it for display! "Its unequal scale divisions make it not well understood by the people.....It is not well adapted to the plotting of vital statistics by months, because it is not the rates of change according to seasons which interests us, but the actual changes (cf. **6.4**)." (p. 87).

	Calcutta	Bombay	Madras	Karachi
1907	22444	12645	1548	530
8	21281	12585	1754	643
9	19776	14375	1948	702
10	22238	16652	2117	755
1	25763	17605	2083	762
2	28831	20831	1152	1159
3	33133	21890	2340	1219
4	28031	17696	2127	1315
5	32266	16462	1887	1352
6	48017	24051	2495	1503
7	47193	33655	2339	2028
8	74397	53362	2528	2429
9	90241	76250	3004	2266
20	153388	126353	7500	3120
1	91672	89788	3847	3579
2	94426	86683	4279	3234
3	89148	75015	4712	4064
4	92249	65250	5546	4515

Is the attitude indicated in the former sentence one to acquiesce in? For the first time in the Census of India this type of graph has been used in 1921: see the very striking graphs in IX, opposite p. 3;[1] also the more obscure graphs in VIII 50, 96, etc.)

Ex. 3. Plot on logarithmic ruling to a suitable scale the accompanying infantile mortality figures. How would the effect of the Notification of Births Act, 1907, be shown on the graphs? Cf. Ex. 4 (*h*),

DEATHS UNDER ONE YEAR PER 1000 BIRTHS.

	Aberdeen	Dundee	Edinburgh	Glasgow	Scotland	England
1900-04	147	163	132	150	122	143
1903	145	154	131	143	120	138
4	139	152	125	136	117	134
5	135	153	126	136	116	131

1 Without detracting from one's appreciation of this graph, it may be pointed out that it is unnecessary to describe the changes of population as "proportional", and that there is no need to show a uniform scale of logs (a guarantee of honesty, is it?); though this gives the example of stationary scales referred to in **1.431**! (Yet one statistician writes, "The logarithmic historigram (**7.31** f.n.), while valuable for relative comparison in point of time, is not good for comparison of the **sizes** of different variables at the same time.....logarithmic curves are of no practical value for showing the absolute size of the different variables at any given date.") The remark, "A slope of 45° means a rate of increase which would double the population in 30 years," should be tested by measurement thus: log2 = 4·95cms. = 30 years on the horizontal scale: cf. Exs. 6, 7.

DEATHS UNDER ONE YEAR PER 1000 BIRTHS— contd.

	Aberdeen	Dundee	Edinburgh	Glasgow	Scotland	England
1906	134	155	126	135	117	129
7	134	149	123	132	114	121
8	130	156	119	130	112	116
9	132	153	118	132	113	115
1908-12	133	155	115	130	112	112
1911	136	157	112	129	110	110
2	131	156	110	129	112	110
3	143	163	113	133	115	111
4	138	158	110	127	112	102
5	140	155	113	128	113	102
6	138	148	112	125	111	100
7	137	146	113	122	109	97
8	128	130	106	117	99	92
9	127	128	105	116	97	90
1918-22	126	124	99	114	96	86
1921	118	117	97	109	92	81
1920-24	119	115	90	110	91	77

These figures are taken from a graph on ordinary ruling at the end of "A Social Survey of the City of Edinburgh". (The figure is an excellent example of how lines may be distinguished from one another, but it is not suitable for reproduction here.)

The following note is attached to the diagram: "The figures on which this Chart is based have been obtained from the Annual Reports of the Registrar General for Scotland. Quinquennial average rates have been used in place of annual rates because the latter are so largely affected by outbreaks of epidemic disease and by abnormal climatic conditions that they tend to vary considerably from year to year. Such variations obscure the general trend of the figures. The quinquennial average rates minimise the effect of these variations, and are thus more suitable for use in showing graphically the broad course of infantile mortality over a period of years." Cf. **6.5, 2.311.**

Ex. 4. Plot semi-log graphs of the figures given in Table III.[1]

The columns refer to deaths due to different causes; the characteristics to be noted are indicated in each case as follows:

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1 The critical student who compares the beginning and the end **(3.151)** of some of the columns will be able to detect the fact that these figures were taken from curves. The curves are published by Whipple and Hamblen in the Reports of the U. S. A. Public Health Service (**37** 1981).

(*a*) **General death rate.**[1] Compare this with each of the other curves. The figures from 1851 to 1869 are

	18,	17,	18,	19,		18,	17,	18,	17,	16
18,	19,	18,	23,	24,		22,	17,	16,	18,	17.

Note the steady general rise to 1890, and the more rapid fall thereafter. Connect features in this and the other curves with events in the following table:

1861—5	American Civil War.
1867	Pasteur, professor of chemistry at the Sorbonne.
1882	Tubercle bacillus discovered by Koch.
1886	The Pasteur Institute opened for the study of bacteriology.
1889—90	Studies of water-purification and sewage-disposal at Lawrence. Free distribution of diphtheria antitoxin. Establishment of tuberculosis dispensaries and sanitoria.
1910	Pasteurisation of milk adopted extensively.
1918	Influenza year.
1920 (Jan)	Prohibition Amendment effective.

Put marks on the time axis to correspond with these and other events which you think relevant. You will then be doing what students of history do who wish to make vivid the relation of events in time. Cf. Keatinge, "Teaching of History", p. 141 (A. & C. Black): also **2.33, 7.531.**

(*b*) **Tuberculosis** (pulmonary). Note the change in rate about 1885. Show the possibility of the **prediction** from this semi-logarithmic graph that by 1950 the deathrate from this cause will be 38 if the present decline continues. Cf. Whipple, "Vital Statistics," pp. 369, 370.

(*c*) **Typhoid Fever.** This graph has been noted above. Which is decreasing more rapidly, tuberculosis or typhoid fever? Cf. "Medical Biometry", p. 125, where the reference is to all forms of tuberculosis; also Whipple, *op. cit.*, 387 (Massachusetts only).

(*d*) Diseases of the **Digestive System.** In 1900 there were changes in classification.

(*e*) **Diphtheria.** Can you determine an approximate period for the characteristic recurrences of this disease? Cf. Whipple, *op cit.*, 378. Note that the break in the downward trend about 1910 may mean that control has been attained, while extermination is impossible. This suggests a possible economy in public health activity.

(*f*) **Scarlet Fever.** Here there is a steady general reduction, "though the bacteriology of this disease is not well understood. This line differs from the diphtheria line in showing no reduction in the regular recurrences". In view of the vertical scale being logarithmic is this comment justified?

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1 In "Vital Statistics," Whipple says (p. 268): "A general death-rate, or gross death-rate, is of little use until it has been analyzed. The 'Total solids' in a water analysis gives the chemist almost no idea of the quality of the water..... A general death-rate must be broken up into its constituent parts..... Death-rate analysis today is in about the same condition that water analysis was in fifty years ago."

TABLE III—Death-Rates in the U.S.A. from 1870 to 1920.

Year	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)
No. per	10^3	10^6	10^7	10^6	10^6	10^7	10^7	10^3	10^6	10^7	10^7
1870	18	340	860	68	47	480	190	160	36	62	51
1	18	335	740	70	50	580	80	151	37	81	58
2	23	360	1050	76	47	850	280	195	36	76	88
3	22	345	830	77	56	920	120	178	39	74	69
4	18	320	700	77	85	830	100	162	37	72	66
5	22	340	660	81	135	1000	150	175	36	93	59
6	20	315	510	77	200	760	65	154	40	73	51
7	18	318	480	78	182	280	29	152	38	98	31
8	18	308	390	79	145	240	180	150	48	78	38
9	17	298	360	80	138	490	11	147	50	92	39
80	20	305	470	96	140	350	130	160	52	74	61
1	21	320	600	95	139	210	125	164	52	90	68
2	20	310	590	100	118	180	38	163	53	88	68
3	21	315	460	105	96	300	180	158	55	88	60
4	19	300	460	102	87	320	39	160	57	97	66
5	20	302	400	106	87	300	170	157	57	91	63
6	18	304	410	100	78	170	65	155	56	74	54
7	20	290	430	107	78	290	220	160	58	80	53
8	20	275	430	107	88	280	100	160	61	78	60
9	19	260	400	109	100	80	75	158	62	80	57
90	19	265	370	108	72	90	52	165	63	80	68
1	20	245	350	115	61	110	100	158	62	74	80
2	22	250	350	121	65	290	38	159	61	102	83
3	21	234	310	122	64	330	110	160	65	103	86
4	18	218	300	119	72	200	40	156	65	92	60
5	19	220	270	119	70	150	47	152	70	101	71
6	17	213	280	122	68	100	54	155	70	101	76
7	16	206	230	120	63	130	60	147	68	88	53
8	16	199	240	125	25	60	31	150	71	100	60
9	17	192	220	120	37	100	85	148	68	96	62
1900	18	185	220	139	51	110	110	154	71	92	80
1	16	175	190	192	39	130	60	140	73	108	62
2	15	165	180	190	30	110	110	142	74	88	59
3	16	155	170	192	30	180	78	141	77	104	67
4	15	164	160	191	23	45	55	137	80	100	55
5	16	156	170	204	21	40	60	142	82	98	60
6	15	148	160	195	23	43	66	140	84	91	52
7	17	150	130	198	23	90	50	136	85	125	70
8	16	138	160	194	22	110	100	139	86	130	35
9	15	133	120	190	18	80	49	128	86	122	57
10	16	133	130	202	20	75	70	132	90	124	64
1	15	128	90	180	15	50	46	118	92	123	60
2	14	120	76	170	14	34	80	119	92	131	56
3	15	114	78	160	18	80	90	114	94	130	94
4	14	113	70	150	18	70	41	109	97	129	80
5	13	112	53	138	19	50	40	105	98	132	50
6	15	121	46	136	17	32	103	101	104	118	57
7	14	130	47	142	21	31	95	97	110	122	56
8	20	140	40	140	16	20	130	110	111	120	30
9	13	110	26	100	16	35	49	98	110	116	18
20	15	98	25	106	15	53	90	86	115		16
1	11	85	30		16		45	90			
2		82									

(*g*) **Measles.** "Not yet successfully controlled."

(*h*) **Infant mortality.** A steadily increasing rate of improvement since 1890. "General efforts in the direction of infant welfare have doubtless had their effect in causing the downward rate to accelerate, but sanitation seems to have been more effective than hygiene." Can you justify this statement? cf. (*c*) and **9.31.**

(*i*) **Cancer.** A contrast! A modern disease?

(*j*) **Suicide.** Earlier figures from 1857 are 80, 70, 70, 89, 74, 74, 52, 51, 62, 56, 57, 64, 65.

Find out the years of financial panic in the U. S. A. and compare them with this graph. (Whipple gives these years as 1857, 1873, 1893, 1907, and connects them with subsequent decreases in the marriage-rate.)

(*k*) **Alcoholism.** Earlier figures from 1861 are
98, 99, 130, 120, 45, 31, 27, 53, 57.

Figures for recent years are difficult to obtain. For 1921,–2,–3 the deathrates for this cause are given as 18, 26, 32, the last being attributed to the drinking of inferior liquor. Other figures given are from 1914–1917, actual number of alcoholic deaths, 14,270; 1919–1922, 6,345. Check the reliability of these figures. (The population of the U S. A. in 1910 was 91,972,266; in 1920, 105,710,620.)

Ex. 5. A rope laps round a fixed post l turns, the slack end being pulled with a force $M=2$. The force N on the other side was just sufficient to prevent slipping; prove that the law $N/M=2{\cdot}718^{1{\cdot}846l}$ connects the following values of N and l:

l	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$
N	3·17	5·06	7·90	12·68	19·90	32·10	50·12	80·3	125·8	201·5

(Perry's "Practical Mathematics", p. 71: the essential part of Ex. 6 is taken also from this book, p. 79.)

Ex. 6. Show from the following table that the population P of England and Wales follows closely the law $P=10^{7{\cdot}03+{\cdot}00556t}$, t being reckoned in years from 1811 (the 1811 census result to be omitted).

	1811	1821	1831	1841	1851	1861	1871	1881	1891
England and Wales ...	10·164	12·000	13·897	15·914	17·928	20·066	22·712	25·974	29·002
Scotland ...	1·806	2·091	2·364	2·620	2·889	3·062	3·360	3·736	4·026
Ireland ...	—	6·802	7·767	8·175	6·552	5·779	5·412	5·175	4·705
India ...	—	—	—	—	—	—	206·162	253·896	287·315
U. S. A. ...	7·240	9·638	12·861	17·063	23·192	31·443	38 558	50·156	62·948

Calculate P from the formula, find the small differences[1] from it as given by the census returns, and plot these differences as a curve. Show that the probable population in 1901 and 1911 was 33·93 and 38·58 millions respectively.

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1 *i. e.*, the deviations of **2.311.**

Note—The correctness of Perry's forecast and "law" may be judged from the following census figures for 1901, 1911, 1921—

	1901	1911	1921
England and Wales ...	32,527,843	36,070,492	37,885,242
Scotland	4,472,103	4,760,904	4,882,288
Ireland	4,458,775	4,390,219	4,470,000 (estimated).
India	294,361,056	315,156,396	318,942,480
U. S. A.	75,994,575	91,972,266	105,710,620

Test whether the figures for the other countries can be treated in a similar way.[1]

These figures are given "accurately", not because they are required so in order to test the prediction, but in view of a footnote which Perry adds; you should consider how far you agree with the note. Having noted that simple curves go *through* points representing numbers calculated from formulæ, and evenly *among* observed numbers, he continues, "Plotting correct numbers, as of Population, the true curve goes exactly through the plotted points. But there is possibly a simple law complicated by perturbations; in studying the curve which goes evenly among the points we look for the general law. Having it, we search for the perturbation law, if there is one." (Cf. **2.311.**) If not, we must be content with the average error (**6.4**). The chief points here will be elucidated by a reference to Whipple's "Vital Statistics" p. 109, etc., especially p. 188 where a noteworthy graph, of the type described in **7.11**, is given to show the age-distribution of the people of Sweden. "The influences which increase or decrease the numbers of children produce results which flow as waves throughout a long life-term". Note also p. 204 on which is given an example of the use of increments (not differences, as above). Cf. the deviations of **6.52.**

To illustrate the remark in **1.33** about manipulation of units we detail here the **deduction of the formula** given by Perry. To determine directly the slope of a line it is convenient if there is some simple relation between the unit of the logarithmic scale and the equi-spaced ruling. On the blackboard ruling on which this work was done (as shown in figure 2) the logarithmic unit is the side of a square which is divided into 10 equal parts for the time scale. There seems to be no reason why Perry should suggest that the 1811 figure should not be included[1] (*v.* Whipple's "Vital Statistics", p. 208); when all his values were plotted the equation of the line through them was found as

$$\log p = \frac{\log 20 - \log 11}{0\,5} T + \log 11.$$

Here the unit for p is 10^6, and for T 100 years. Accordingly we get $\log(P/10^6) = \cdot 52.\, t/100 + \log 11$ or $P = 10^{6} + \cdot 0052t + 1\cdot 0414 = 10^{7\cdot 04} + \cdot 0052t$. (Alternatively, values of P and t might have been substituted twice in $P = 10^{mt+c}$ i

1 It will of course be recognised that figures of early censuses may not be comparable with those for later, owing to changes of territory, etc.

and the two equations solved for m and c_1.) The populations for 1901 and 1911 were estimated from this graph at 32·5 and 37 millions: working on good semi-log ruling you should attain a higher accuracy.

Ex. 7. The charts in Whipple's "Vital Statistics", p. 205, etc., showing growth of national populations, have straight lines of slope 1·1 per cent drawn across them for comparison. By measurement the actual slope of these lines was found to be 1·68/7·73. The unit of the logarithmic scale is 2·70 cms., and 100 years are represented by 5·95 cms. Verify that the lines show correctly a slope of 1·1 per cent.

(Find the number of years represented by 2·7 cms.; divide the slope by this number to get log $(1+r)$: cf. **9.32**, f.n.)

Ex. 8. Prove that the angle of inclination of a line which shows an increase of r per cent. on logarithmic ruling, in which the time unit is $1/k$ times the logarithmic unit, is $\tan^{-1}\log\left(1+\frac{r}{100}\right)^k$. Apply this to check these measurements from Whipple's reference chart (p. 211), where the scales are as in Ex. 7.

Per cent.	1	2	3	4	5	10	15	20	30	50	100
Angle	11°	21°20′	30°30′	37°40′	43°40′	61°40′	69°40′	74°25′	79°10′	82°30′	85°40′

Modify this formula to make it applicable to decreases. What is the limiting case?

9.4. PARETO'S LAW. The advantage of using semi-logarithmic graph paper for certain purposes leads us to ask if there is any advantage in using graph paper in which *both* the axes are graduated logarithmically. The general equation to a straight line drawn on such paper is

$$\log y = m \log x + \log c, \qquad i.e., \; y = cx^m;$$

and so this type of paper is useful when we wish to represent a relation which involves one of the variables as a power. Any of the **simple parabolic curves** of **2.1** would on this graph paper be transformed to straight lines. Thus, for example, may be obtained easily and rapidly a graph to represent the relations between the area and side of a square, pressure and volume of a gas, and so on.

9.41. Here we shall consider in some detail in its mathematical aspects[1] a simple law dealing with a very complex subject in economics. A well-known Italian econom-

1 These, however, relate only in a minor degree to the graphical representation on logarithmic ruling.

ist, Pareto, in studying the **distribution of wealth** among the members of a community, found by trial that there was a relation between size of income and the number of people who had that income: this was represented by

$$N = A/x^s$$

where N is the number of persons whose income is at least x units (rupees, liras, dollars, etc.), and A and s are constants which depend on the country, or the class of the community, that is being considered. Writing this in the logarithmic form, we have $\log N = \log A - s \log x$, and this we know is represented on logarithmic ruling by a straight line whose slope is negative, s being >0, in fact, >1. Figure 52 is drawn in the way this line appears most frequently in books on economics: to see it as mathematicians are accustomed to see it you must turn the right side uppermost, and then get behind the paper![1]

Mathematically the law is very simple, and we might content ourselves by expressing the wish that economists who discuss it would represent it in some uniform way. It *is* possible to interchange axes (**2.11**) and to vary the relations of the constants; but, unless some definite purpose is served thereby, an unusual presentation of the facts tends to produce confusion, and those who are not experts get the impression that the significance of the facts as to the distribution of wealth is much too recondite for them to appreciate.[2]

9.42. It does not concern us here that some economists doubt[3] the reliability of this "law": *we* are in no position to discuss this. The data economists have to elucidate are even

1 This is a good mental exercise: it might have been employed with advantage in **2.11**. Cf. also **7.31** Ex. 5, note.

2 The formula usually given has α for the index s. Mathematicians habitually denote *angles* in circular measure by Greek letters, and so to use α for a slope involves extra mental effort for them! For convenience in printing, s has been used here for α; it might, however, have been better to have followed mathematical convention completely and taken s as the slope, measured in the usual way (p. 42, f.n. 2): Pareto's law would then have been simply $N = Ax^s$, where s is negative.

3 Yet "While the logical foundation of Pareto's law is open to controversy, there is no doubt about the validity and importance of the law." (F. Y. Edgeworth in Palgrave's "Dictionary of Political Economy", 3 713.)

more elusive than those we have encountered in chemistry or physiology. Besides their difficulties of definition and of getting reliable data, there is this, that it is impossible to conduct experiments in economics in a laboratory where suspected cause and effect may be isolated from other possible influences and directly connected with one another. Experiments can be watched only when they happen to be taking place ; but even then the results cannot be regarded with the detachment that is possible in ordinary scientific experiments. No one can be indifferent to the facts that the political and economic experiments carried out in recent years in Russia, as a result of the Bolshevist Revolution, have involved a reduction of the population of Petrograd from 2,250,000 in 1914 to 700,000 in 1921; of Moscow from 1,800,000 to 1,000,000; and of the whole country from 180 to 130 millions. Even in a readjustment of tariffs and taxes, more or less thorough, few people "detach" themselves from their pockets, even in the interest of the State !

As to Pareto's law, however, it is certain that for extreme positions it is not reliable ; in fact in some investigations the people of a nation are divided into at least three classes, and it is found that the law holds for each of these classes separately, the value of s for the wealthy class being greater than that for the poorer; and so the distribution of wealth may be more truly represented by a series of three straight lines, say, none of which reaches the axes of logarithmic co-ordinates (**2.14** ; cf. fig, 53). This modification of the law of distribution has a very intimate connection with the important distinction between '**earned and unearned incomes** (cf. **6.31**) : but it is difficult to say how far the law as modified is exact.

9.43. It is notoriously difficult to get reliable information about incomes and so it is very important to be able to check the worth of such information as is obtained direct by comparison with say, estate duties, house rent, or claims for exemption from income tax—criteria that were used in the three ranges referred to above. This law has been used successfully[1] to guide income-tax authorities as to the class of income-tax payers among whom there was most evasion of payment—not a very pleasant law from some points of view !

9.44. It should be noted that this graph, when drawn on ordinary ruling, has a relation to the integral curve we

1 Sir Josiah Stamp : " Wealth and Taxable Capacity ", p. 82.

considered in **7.31** f.n., though the number N (or log N) is exclusive in its tendency as x increases; it does *not* include the numbers in the classes lower than the value of x that is being considered. This may be seen more easily by taking the x axis horizontal, as described in **9.41**: the curve then appears as a reversed integral curve—N is the number of people who *have* incomes down to that shown on the x axis as we move from left to right! The matter may be seen still more clearly if we compare various curves thus: in **7.31** f.n, δy is the number between x and $x+\delta x$, in **7.32** this number is y, while here δN is the number between $x-\delta x$ and x.

This suggests, what is actually so, that the information which is given in a **frequency distribution** can be **derived** from this "integral" formula by the aid of the infinitesimal calculus. The number of persons whose income is x is the difference δN (really $-\delta N$, but the application is so obvious that we need not burden[1] ourselves with the sign) of the numbers of those who have incomes $x+\delta x$ and x. Here we take δx as a suitable, comparatively small unit, £1, or Rs. 2 or Rs. 100, and we find (**3.15**) neglecting signs, the number whose income is between x and $x+1$ as $\dfrac{dN}{dx}\delta x = \dfrac{sA}{x^{s+1}}$

If we call this n, we get a relation between income x and the number who have that income (though this time, neither more nor less), and this also can be expressed by a straight line on logarithmic ruling: for

$$\log n = \log (sA) - (s+1) \log x.$$

The only important difference is that the inclination of the line to the income axis is greater: and so Pareto's law might have been expressed in this form.[2] Here again the number n in the successive classes increases as x decreases,

1 This does not mean that signs are usually a burden, and here certainly the burden is not heavy. Signs are always a sure guide, but here we have other guides that most of us know better.

2 The statement of Pareto's law given in the Encyc. Brit. **22** 390d is worth quoting though, to make it quite consistent, "at least" should be added before the last two words: "The number of incomes of different sizes (above a certain size) is approximately represented by the equation $y = A/x^{\alpha}$, where x denotes the size of income, y the number of incomes of that size." Reasons for preferring this form of the law to that as a frequency distribution are given under Ex. 4.

and we see that the frequency curve, drawn on uniform ruling, is completely skew (cf. a remark at the end of **7.42**). Ability, like other natural qualities would be represented by a nearly symmetrical frequency distribution; the **skewness** of the frequency curve for income corresponds to the fact that income depends upon accidents of inheritance, and not simply upon ability. (Cf. Pigou, "Economics of Welfare", p. 696.)

Ex. 1. Show that the ratio of the numbers of those who have at least a certain income to those who have just that income varies as the income; in fact $nx = Ns$.

Ex. 2. The distribution of incomes on which super-tax was paid in Britain in 1911-2 was as in the accompanying table. (Note how the irregularity of the classes would make the points representing the figures directly difficult to interpret.) Modify the second column so as to show the numbers of incomes not less than the several sums stated, and plot logarithmically the points showing these results. Find the values of A and s in the formula $N = A/x^s$ corresponding to this distribution.

Income (£1000)	Number Recorded.
5 -	7,411
10 -	2,029
15 -	787
20 -	438
25 -	382
35 -	186
45 -	107
55 -	56
65 -	37
75 -	55
100 -	66

Bowley in his "Statistics", p. 347, gives the values $s = 1{\cdot}5$, $\log A = 9{\cdot}618$; these have probably been obtained by arithmetical methods (cf. p. 75, f.n.). In his paper in the Quarterly Journal of Economics **28** 255ff. he considers the possibility of these figures, along with figures from income-tax abatement, being represented by two parallel Pareto lines of slope 1·5, for which $A = 9{\cdot}30$ and $9{\cdot}62$ respectively.

Ex. 3. In King's "Statistical Method" p. 99 the following, apparently hypothetical, figures are given of the numbers of men with incomes in 1000 dollar classes up to 18,000. Show that no Pareto line fits these figures.

5, 8, 10, 12, 14, 10, 9, 10, 6, 2, 3, 1, 2, 0, 2, 1, 1, 1, and 3 above 18,000 dollars.

Ex. 4. In Yule's "Statistics", p. 83, the accompanying figures of dwelling houses assessed to Inhabited House Duty in 1885-6 are given as an example of the difficulty caused by unequal frequency intervals. Find the Pareto formula which fits these figures for most of the range.

Annual Value	No. of Houses
£20 –	306,408
£30 –	182,972
£40 –	105,407
£50 –	63,096
£60 –	71,436
£80 –	32,365
£100 –	41,336
£150 –	26,732
£300 –	6,198
£500 –	2,098
£1000 –	644
	838,692

Note how these figures bring out **the convenience of plotting total numbers** above each value: (i) There is no need to plot at the middle of intervals of unequal size, (ii) the difficulty of vague end-intervals is avoided (**7.311**), (iii) the range in the number-scale is reduced——here the frequencies for a £10 interval range from 10 to 10^6, and so require an extra logarithmic unit.

Ex. 5 Another example of a *J*-shaped distribution is given by Yule (*op. cit.* p. 100). The figures are of historical interest, for they refer to the estates of those who took part in the Jacobite Rising of 1715. Show that, though the figures are said to be "of very doubtful absolute value", they lie very closely on two Pareto lines for the lower and higher parts of their ranges respectively; find the values of s for the two "laws". (The numbers at the end of the table are really within the unit classes indicated, *e. g.* 2 in 27-28; but this makes no difference if you plot N, and not n.)

Annual Value (£100)	No. of Estates
0 –	1726·5
1 –	280
2 –	140·5
3 –	87
4 –	46·5
5 –	42·5
6 –	29·5
7 –	25·5
8 –	18·5
9 –	21
10 –	11·5
11 –	9·5
12 –	4
13 –	3·5
14 –	8
15 –	3
16 –	5
17 –	1
20 –	4
21 –	1
22 –	1
23 –	1
27 –	2
31 –	1
39 –	1
45 –	1
48 –	1
	2476

N. B. Four logarithmic unit ranges are required for the plotting of these figures, but different scales may be used for the same units without confusion, *e. g.* the lowest unit may show nnmbers both from 100 to 10 and 10 to 1, or the graph may be drawn on two units showing both 10,000 to 100 and 100 to 1; parts of the same Pareto line thus separated will appear parallel.

Ex. 6. Show that wages statistics. *e.g.*, those given by Bowley, *op, cit.* p. 69, cannot be represented by a Pareto line. Assign reasons for this.

Note—Attention should be drawn to the material in Stamp's "British Incomes and Property" (P. S. King & Son, 1920), which may be investigated with the help of logarithmic ruling; *e. g.*, pp. 36, Graphs of assessments should be re-drawn on semi-logarithmic ruling: p. 333, Pareto line of Ex. 2 is plotted, and for it $s=1{\cdot}66+$ is found; p. 338, Three sets of super-tax statistics for which s is almost 1·75. Cf. **9.42**; p. 457, An interesting diagram showing a relation between rent and income; the figures of each of these agree with a Pareto formula; p. 518, Numbers of houses assessed to house-duty in 1829 in England and in Scotland: each set of figures lies along two Pareto lines, the respective slopes in the two portions of their ranges being distinctly greater for Scotland than for England. Approximate values of s for England are 1·2. 2·2; for Scotland 1·6, 2·4.

9.441. The comparison of the graph for N on ordinary ruling with an integral curve can be put positively in

another way than that suggested above. Suppose the graph viewed from behind the paper; the reversed horizontal income axis may then be regarded as a positive **poverty scale.** The ordinate of the curve, which rises to the right, represents the number of people who have at least as much poverty as is indicated by the reversed abscissa! Cf. figure 54. For the ogive, which is got simply by turning this figure, viewed from in front, clockwise through a right angle, the corresponding statement is: an ordinate represents poverty which has not been attained by the number of people represented by the abscissa.

The typical form of the integral or cumulative curve is simply $\int$ (Bowley, *op. cit.*, p. 106; Pearl, *op. cit.*, p. 118). Comparison of this with figure 54, viewed from behind, suggests that the departure from Pareto's law in the region of small incomes may have **a natural explanation** behind that suggested in Palgrave's Dictionary, III 712. "As the law relates to *averages* it is notto be expected that it should be verified at the higher extremity where only one or two observations occur.....With regard to the lower extremity of the curve, the shape depends on our definition of "income": whether with Pareto we include paupers, or restrict the definition to a more homogeneous class." But this is a question for economists. In any case, the contrast between the departure at the large-income end of our curve and that at the lower end of the "natural" curve is striking, and appears to confirm the comment at the end of **9.44.**

The most obvious graphical comparison, however, is with figure 10, where ordinates represent the numbers not yet dead in a time as short as x; in figure 54 an ordinate represents the number whose incomes are not as small as x. Comparison of figure 10 with the ogive or with the integral curve may provoke many reflections as to how **the natural course** of events may be **altered by man.** A sentence in Palgrave's Dictionary will extend still further our thoughts on this subject, and on the interpretation of the form of curves: "While dimly discerning that universal statistical principles and stable human institutions are behind the Paretian formula, we need not assume such fixity of causation that

the inequality of distribution *cannot* be altered so long as the magnitude of the aggregate dividend remains unaltered."

Ex. 1. What effect would the inclusion of paupers among the income-earners have on the Pareto curve?

Ex. 2. Does the absence of agricultural incomes from the Indian income-tax returns make these unsuitable for treatment by Pareto's law?

Ex. 3. A percentage-wealth axis reversed (*i.e.* a poverty axis) and a percentage-population axis have been used as a basis for curves representing the distribution of property. Show that equal distribution is represented by a straight line, and discuss the properties of the figure. (King, *op. cit.*, p. 156.)

9.45. Sometimes economists have to reason from total figures, or at least to check their reasoning by reference to some **convenient aggregate**, *e.g.*, the total income assessed for income tax. By integration such a total may be obtained from Pareto's formula: thus, since every number N of people receives income $N\delta x$ above Nx, the aggregate income of those earning more than x units is $Nx+\int_x^\infty N\,dx$. But

$$\int_x^\infty N\,dx = A\int_x^\infty x^{-s}\,dx = A\left[x^{1-s} \Big/ (1-s)\right]_x^\infty \quad (3.212)$$

$$= A\left[0 - x^{1-s} \Big/ (1-s)\right] = \frac{A}{s-1}\,\frac{1}{x^{s-1}}$$

for s is always found to have a value greater than 1, usually about 1·5. And so the required aggregate income is

$$\frac{A}{x^{s-1}}\left(1+\frac{1}{s-1}\right) = \frac{s}{s-1}\,A\frac{1}{x^{s-1}}$$

This could have been obtained directly from Pareto's law expressed as $n = sA/x^{s+1}$; for

$$\int_x^\infty xn\,dx = As\int_x^\infty \frac{dx}{x^s} = \frac{As}{1-s}x^{1-s}\Big]_x^\infty = \frac{As}{s-1}\,\frac{1}{x^{s-1}}$$

Similarly $\int_x^\infty n\,dx$ may be found to be A/x^s which is simply the number N (**7.3**). So also may be found the total income in a range x_1 to x_2, etc.

Ex. 1. Show that under Pareto's law the average income above £x is £$\frac{s}{s-1}\,x$; and that, if $s=1{\cdot}5$, the average income in the interval £x_1, to £x_2 is

$$£\,\frac{3x_1\,x_2}{x_1+\sqrt{x_1x_2}+x_2}.$$

(The three following examples are from Perry's "Practical Mathematics", p. 75).

Ex. 2. Show that a law of the form $P = 0{\cdot}346\ V^{1{\cdot}69}$ runs through the results noted below of experiments in towing canal boats :

Pull, P lbs per ton	0·70	1·70	2·35	3·20	3·50
Speed, V miles per hour	1·68	2·43	3·18	3·60	4·03

Ex. 3. A law of the form $y \propto x^m z^n$ can be calculated in two stages, first keeping x constant and then z constant. Thus the formula $I = 0{\cdot}00513\ D^{\ 0{\cdot}59}\ v^{\ 2{\cdot}97}$ fits the following measurements on a steamer.

I. H. P.	140	410	820	1500	442	351	294
Displacement	1748	1748	1748	1748	2030	1400	1000
v knots	7	10	13	16	10	10	10.

Ex. 4. Values on the expansion curve of the indicator diagram (**3.221**) were measured as follows

p	39·6	44·7	53·8	73·5	85·8	113·2	135·8	178·2
v	10·61	9·73	8·55	7·00	6·23	5·18	4·59	3·87.

It was known that the clearance was not measured exactly ; hence v needed a constant correction. Prove that the formula that fits these values is

$$p\ (v - 0{\cdot}6)^{\ 1{\cdot}346} = \text{constant}.$$

(Before you plot these values, make up your mind which way you expect the line through the points will curve on logarithmic ruling.)

Ex. 5. In an investigation of a cholera epidemic in Poona, a connection was found between the number of deaths in the city and the rainfall in the catchment area nine days earlier. The figures are

Daily average rain, R	0	·123	·356	·744	1·467	2·874	5·358	8·290
Daily average deaths, D	·24	·33	·61	1·19	2·27	4·625	6·545	14·000

Prove that the relation between these is approximately $D = 2R^{\ 0{\cdot}81}$.

By your result test the statement, " 100 per cent increase in the rainfall was followed by 88·5 per cent increase in deaths." (Indian Journal of Medical Research **4** 68),

(The averages in this example were got by counting the numbers of days on which the rainfall was within intervals, each of which, save the end " intervals ", was double the interval below it, *i. e.*, the upper boundary-values were ·24, ·48, ·96, 1·92,.....Can you suggest a reason for this ? The numbers of days were 29, 21, 18, 21, 11, 24, 11, 1. The actual rainfall total for these days, and the numbers of deaths on the days nine days later, were used to get the averages stated. These can be checked from the figures on p. 102, where, however, the total number of days is 147.)

9.46. MORTALITY FORCE : Pareto's law is one instance of the relative rate of change (**3.152**) being expressible in terms of the independent variable ; for

$$\frac{1}{N}\frac{dN}{dx} = -\frac{x^s}{A}\ Asx^{-s-1} = -\frac{s}{x};\ \text{similarly,}\ \frac{1}{n}\frac{dn}{dx} = -\frac{s+1}{x}.$$

Other cases of this we have had in the first two of Perry's rules given in **1.33**, the second of these being really the same as Pareto's law. In Bowley's "Statistics" II Chap. V are given several expressions of this type, of which Pareto's formula is one. The most general case

$$\frac{1}{y}\frac{dy}{dx} = \frac{x+a}{b_0+b_1 x+b_2 x^2}$$

is somewhat too difficult for us to investigate; but we can consider the significance of the **Gompertz—Makeham formula**

$$-\frac{1}{y}\frac{dy}{dx} = a+bc^x,$$

which represents the number y of a given generation who survive to the age x, and a, b, c are constants. This is what we have represented for us in figure 10, which we have already tried to investigate. The relative death-rate shown by this formula is called the *force of mortality*: it is composed of a **constant** part a, signifying the **risk** of death common to people of all ages, and of a term which represents the **increasing risk** with advancing age as changing geometrically (**2.21**).[1]

In **2.131** the suggestion was made, merely on the basis of the appearance of the curves, that the equation $y+(x-1)^n=0$ might fit the curves of figure 10, for parts at least of their range. To decide how far this was justified, consider $Dy/y=\{-n(x-1)^{n-1}\}/\{-(x-1)^n\}=-n(1-x)^{-1}$, for x was taken less than 1. This, for the same reason (**1.341**), may be written

$$Dy/y=-n-nx-nx^2-\ldots\ldots\ldots\doteq-n-nx;$$

and the resemblance to Makeham's formula is apparent. The chief difference is that the risk which increases with age is represented as increasing arithmetically—a reasonable enough assumption, made by De Moivre in the early days of the study of life contingencies. It should be noted also

1. Cf. Institute of Actuaries' Textbook, Part II, pp. 70 ff.

that it is a defect in this parabolic formula that it has only one arbitrary constant; the curves that may be represented by this formula cannot intersect like the survivor curves,

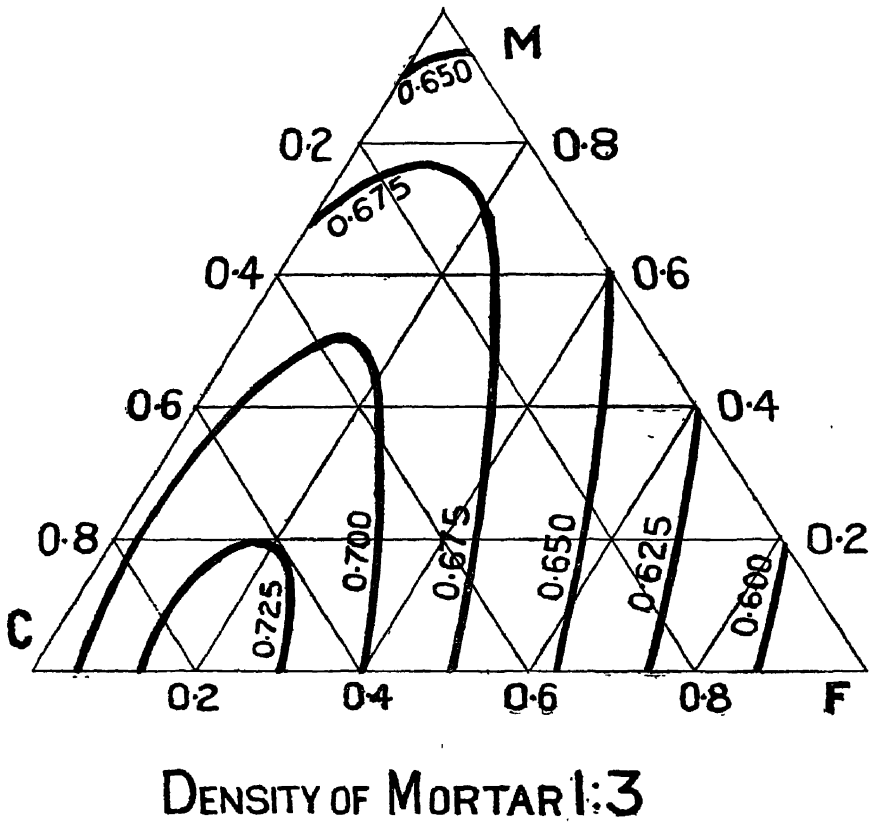

Fig. 65. Variation in the weight of mortar according to the proportions of fine, medium and coarse sand.

unless the unit for the independent variable is made arbitrary, and is taken of different magnitudes in different cases.

Ex. 1. Connect the above similar expressions for the proportional rate of change of x and N in Pareto's law with the formula $nx = Nx$. (9.44 Ex. 1)

Ex. 2. Show with the help of **3.22** ii that the integral form of Makeham's formula may be written

$$y = kh^x g^{c^x}.$$

Ex. 3. Consider the form of the curve in which relative change varies as x—the case in the general form given above in which $a=0=b_1=b_2$. (Writing b_0 as $-\sigma^2$ gives us the usual formula for the *normal curve of error* referred to in **7.55**, etc.)

Ex. 4. Show that Pareto's law is the particular case of the general formula in which $b_0=0$ and $b_2=b_1/a$, $=-1/(s+1)$ or $-1/s$ according as y means n or N.

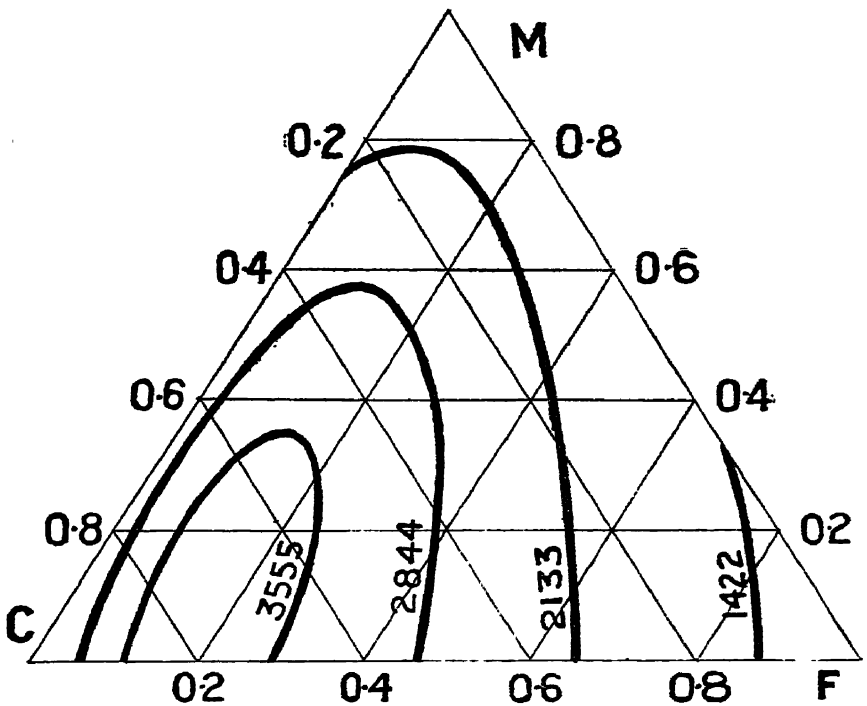

Fig. 66. Effect of varying ratios of fine, medium and coarse sand.

9.51. TRILINEAR CHARTS: Frequently problems arise in which we have to deal, not with two variables only, but with three whose sum may be expressed as a constant,

e.g., the quantities of the constituent metals in alloys, the proportions of elementary gases in mixtures of different explosive powers. The treatment of these questions is greatly facilitated by the use of the simple geometrical fact that the sum of the perpendiculars on the sides from any point within an equilateral triangle is constant. (For the whole area is the sum of the areas of the three triangles with common vertex at the point and bases the sides of the triangle.) The length of the perpendicular from a vertex to the opposite side is taken to represent 100 or 1, and then the distances of any point within the triangle from the sides give the percentage or fractional quantities of the three components in accordance with some specified arrangement. Thus the three *vertices* represent 100% of each component, i.e., a pure substance without any mixture; any point on *a side* represents a combination of two components only; and so on. Conversely, the composition of a substance which has been analysed in a specified way into **three components** can be **represented** by a point, and by that point only. Further, this point may be regarded as the intersection of two lines; for points on a line parallel to one side of the triangle represent substances which contain a constant percentage of the corresponding component, and the intersection of two such lines, each for one component of the substance, gives the required point which represents the substance.

9.511. The accompanying figures, 65, 66,[1] represent properties of cement mortars which are composed of cement and sand in the ratio 1:3. The immediate purpose of the diagrams is to show the effect of using varying quantities of sand of three different degrees of coarseness on the weight and strength of the resulting mortar. The sands used are distinguished as **fine, medium** and **coarse,** denoted in the diagrams by their respective initial letter: the meshes of the screens used in sifting such sand may be, say, 0·5mm. (to reject dust), 2mm., and 5mm. To represent the

1 (Taken from Ira Baker's "Treatise on Masonry Construction", p. 113, John Wiley)

component weights of these sizes of sand in various mortars, sets of lines are drawn parallel to the sides. These correspond to the perpendicular rulings of ordinary graph paper, and so we get a network of triangles instead of a network of squares. These parallel lines may be graduated at any convenient position in their lengths: the device used in these figures is to place the letter at the vertex, beside the line on which are the graduations for the corresponding kind of sand; and so the letter designating a component is close to its highest graduation.

The use to which such diagrams may be put is shown very strikingly in this case. By experiment the densities and strengths of specimens of mortar are found, and these values are marked in the positions which represent the proportions of sand in the specimens. It is then found that equal values of these **properties** of the mixtures lie in their respective diagrams **along curves**. These happen to be somewhat alike in these two diagrams, which is reasonable in this case; and it is easy to see from them that the heaviest and the strongest mortar can be obtained by using no medium sand and by mixing coarse and fine in the ratio 4:1

9.512. The **interpretation** of this is interesting: it is easily seen that by combining material of different sizes the interstices between the strong larger pieces are filled up and a closer binding together of these is made possible than if the pieces were uniform in size. But it could not have been foreseen that the grading of the sizes of sand should not be continuous for the best results, and the proportions in which different sizes should be used could have been only a matter of guesswork. The diagram by giving easily so clear a meaning to results shows that experiments to test mortars are consistent enough to be worth while; and it gives the satisfaction of knowing that the very best effects have been secured in the conditions specified.

A generalised figure, 55, is given to suggest how the curves in another case might run, not unlike the **contour lines** of a hill: the summit would then represent the best (or worst) proportions of the components. This figure and the form of the curves in figures 65 and 66 suggest that the next experiment to be tried should be to determine whether the addition of some still finer sand would not give an even heavier and stronger mortar.

Ex. 1. Describe the changes in composition of mortar that are represented by a point moving along, say, the ·7 and the ·65 lines of figure 65.

Ex. 2. In figure 65 mortars of the same density are represented as containing two different ratios of the same sands, *e. g.*, a density ·7 is obtained when fine and coarse sands are used in the ratio either 5 : 95 or 41 : 59. How is this possible?

(For the most closely packed mortar the spaces between the grains of coarse sand, " voids " they may be called, occupied by fine sand, amount to ¼ of the space occupied by the coarse sand itself. If there is not enough fine sand to fill the voids, the antecedent of the ratio,

quantity of material : volume of the body,

is less than it might be ; if there be more than enough fine sand to fill the 20 per cent voids, the surplus cannot find dense particles between which to pack, and so the consequent of the above ratio is greater than it need be: in both cases the ratio which gives the density is less than it would be if the kinds of sand were associated in the ratio in which they pack best. Note that there are at least five things to consider in dealing with the properties of this mortar—air and cement as well as the three kinds of sand.)

Ex. 3. Determine what information, beyond what is given in the graphs, would be required if the reasoning of Ex. 2 were to be checked quantitatively. (The measurements of sand are by volume.)

9.513. What has been indicated above with respect to lines through the vertex may be **tested experimentally.** If triangular wedges of glass shaped so that they fit together into a not too thick prism of triangular section, are coloured with the primary pigment colours,[1] blue, yellow, red; then over the section of the prism seen by transmitted light should appear all possible tints due to the combining of the colours in all possible proportions. If only two of these wedges are combined say, the blue and the yellow, as indicated in figure 56, then the ratio of blue to yellow along any line through the vertex O is constant, and the resulting green tint will be uniform along that line; though there will be a difference in intensity according to the thickness of the glass the light has to pass through.

Ex. Carbohydrates contain carbon, hydrogen and oxygen, the two latter in the ratio by weight 1 : 8. Show that all carbohydrates are represented by a straight line in a trilinear chart. On what line of this chart would hydrocarbons (which contain only carbon and hydrogen) be represented ?

9.514. It is not difficult to see how arithmetical calculations to determine the best combinations of **foodstuffs** for a specified purpose may be simplified by the geometry of the trilinear chart. The process will be found exemplified in Haskell's " How to Make and Use Graphic Charts ", p. 32, (Codex Book Co., New York.)

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1 *v.* Webster's Dictionary, *s. v*, " color ".

Ex. 1. The **percentages of heat** produced by the three chief constituents of different foodstuffs when consumed is shown in the following table. Represent the heating values of these foodstuffs by points on a trilinear chart.

Food :	Rice	Dal	Milk	Bread	Butter	Chicken	Oyster
Carbohydrate ...	90	70	29	81	0	0	29
Protein ...	9	27	19	14	0	81	19
Fat ...	1	3	52	5	100	19	52

Ex. 2. If in health a person should receive nourishment from carbohydrate, protein, fat in the percentages 55, 14, 31, find from a trilinear diagram in what percentages protein and fat should function, if no carbohydrate were given and the relation between the other constituents were maintained.

Ex. 3. Represent graphically the **composition by weight** of the foods mentioned in the accompanying table of food values important in the treatment of diabetes. (Joslin, " Treatment of Diabetes Mellitus 3, p. 422.) The figures are grammes weight of the constituents contained in 30 gms of the foods. Note that the necessary modification of the figure can easily be effected by the slide rule.

(The difference between the sum of the weights and 30 is due to the presence of water and other substances.)

Food (30 gms.)	C	P	F	cals
Oatmeal (dry) ...	20	5	2	118
Shredded wheat ...	23	3	0	104
Cream, 40% ...	1	1	12	116
Cream 20% ...	1	1	6	62
Milk ...	1·5	1	1	19
Butter ...	0	0	25	225
Brazil nuts ...	2	5	20	208
Oysters, six ...	4	6	1	49
Meat (cooked, lean) ...	0	8	5	77
Chicken (cooked, lean)..	0	8	3	59
Bacon ...	0	5	15	155
Fish ...	0	6	0	24
Cheese ...	0	8	11	131
Egg (one) ...	0	6	6	78
Cucumbers, tomatoes cabbage, etc. ...	1	0·5	0	6
Turnips, carrots, onions, etc. ...	2	0·5	0	10
Potato ...	6	1	0	28
Bread ...	18	3	0	84

Ex. 4. The food values of the stated amount of foodstuffs in the above table are given in calories. Determine from the table the multiples of the grammes of carbohydrate, protein, fat which have to be used to convert them into calories of food value. Represent graphically the heat producing capacity of each food. (Letter neatly each point appropriately.) Make a statement comparing the resulting diagram with that of Ex. 3. Also compare the diagram with that for Ex. 1, the data for which were taken from other sources. (Note the remark after Ex. 3 in **4.14**).

Ex. 5. Draw on the figure of Ex. 3 the curves which represent foodstuffs which give equal numbers of calories for given weights of the three constituents. What represent these curves on the figure of Ex. 4 ?

Ex. 6. Represent graphically as many as possible of the facts given in this table of variations in diet according to race. (Joslin, *op. cit.*, p. 417).

Race	Weight kilos.	Carbohydrate gm.	Protein gm.	Fat gm.	Total calories.
Eskimo ...	65	52	282	141	2604
Bengali ...	50	484	52	27	2390
European ...	70	512	118	65	3055
American ...	70	400	100	100	2900

9.52. DIET ADJUSTMENT: A diagram is reproduced as figure 58[1] which is suggestive of a trilinear chart, though the triangle is not equilateral. It is connected with the problem of discovering how the body metabolism is functioning—in what proportions protein, carbohydrate, and fat are being used, and in what proportions they contribute the energy produced (cf. **9.514** Exs. 2, 3). For this twofold purpose two quantities are, as in **9.2**, computed and marked against two uniform, suitably adjusted scales at right angles. A study of the figure itself will show its use, this being indicated by the lightly-dotted lines.

9.521. (i) N stands for the grammes of urinary nitrogen excreted in a given period, O_2 for litres of oxygen consumed during the same period. The ratio of nitrogen excreted in urine to the total amount of oxygen used in metabolism is shown on the vertical scale marked N/O_2. The nitrogen comes from the protein consumed; and so at once, from these two *primary* measures, O_P, or simply p, the percentage of the oxygen referred to above, which is used in the **oxidation of the protein** in the food, can be determined. This is marked on a scale alongside the N/O_2 scale. Both scales are uniform, and the range of the N/O_2 scale indicates that if the whole of the oxygen could be used for the consumption of nothing but protein, then 16·83 gms. of nitrogen would be excreted for every 100 litres of oxygen used.[2]

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1 From Dubois' "Basal Metabolism" p. 73.

2 This agrees with the 16·28 gms. nitrogen given by Du Bois on p. 37 as excreted when 100 gms. of meat protein are consumed; for the oxygen required for this is 96·63 litres, *i.e.*, 100 litres of oxygen correspond to 16·83 gms. of nitrogen These two scales for N/O_2 and O_p are another example of stationary scales (**5.5** f. n.) many more occur in figure 60.

(ii) Along the horizontal axis is marked on a uniform scale the other computed quantity, the **respiratory quotient,** *R.Q.* This is "a measure of the kind of material burnt in the body,"[1] *i.e.*, it is a number by which we can tell in what proportion any two of carbohydrate, fat and protein are using up the oxygen being consumed by the body, when the percentage used by the third (in practice, protein as above) is known. The *R.Q.* is found by measuring the amount, and finding the composition of, the air exhaled; hence can be found the quantity of oxygen used in metabolism and of carbon dioxide given off: the *R.Q.* is taken to be the ratio, volume of CO_2: volume of O_2. Chemists tell us that if we were consuming only carbohydrate this quotient would be unity; if we could assimilate only fat, it would be only 0·707; while if we were to get all our energy from protein (which we never can) it would be 0·801. Thus by combining fat and protein we could get any respiratory quotient between ·707 and ·801; and so for the other pairs.

A grid for the *R.Q. i.e* a set of parallel lines through the graduations of an *R.Q.* scale is fitted in an interesting way into the blood-system chart (**9.2**) given in the Journal of Biological Chemistry **59** 396 (fig. 64). This grid is analogous (save that the *R.Q.* scale is uniform, not segmentary) to the simultaneous equation nomogram referred to in **5.1** f.n. (cf. **5.221** Ex. 3); for the lines in this grid are parallel to the parallel scales on which are shown the volumes (or equivalent measures) of O_2 and CO_2. Cf. p. 85, f. n.

9.5211. Accordingly, if **any triangle** were marked out with its vertices *C*, *P*, *F* in positions corresponding to the respective graduated values of the *R.Q.* (though at any distances from the scale), these vertices could be taken to represent the oxidation (or metabolism) of nothing but carbohydrate, protein, fat respectively; and, as in **9.51,** it is clear that by drawing **equally-spaced parallels** to each of the sides the oxidation of any combination of these constituents of food may be represented,—in pairs along the sides, in threes by any point within the triangle. (Fig. 67.)

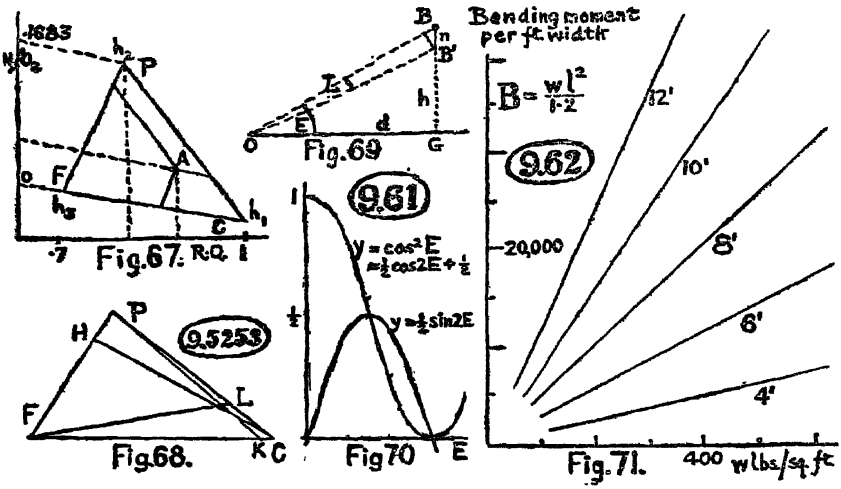
1 "Lancet" **199** 1043, where an excellent account is given of the metabolic rate and its significance in the treatment of diseases like goitre.

Following in the main[1] the **notation** given by Michaelis in the Journ. of Biol. Chem. **59** 56 we shall denote by c, p, f the percentages of oxygen consumed by carbohydrate, protein, fat respectively;

C', P', F' the percentages of the calories derived from each; and

γ, π, ϕ the percentages of each in the metabolic mixture.

The construction for getting the percentages of oxygen used as represented by any point A, is: through A draw (or suppose drawn) a line to the c scale parallel to PF, to the



p and f scales lines parallel to FC, CP. Then the distances of F, C, P from the respective intersections give the required percentages. The cyclic symmetry in this should be noted.

Ex. Convert the essential part of **9.51** from a statement about perpendicular distances into one in this form.

In any such figure the N/O_2 scale must be placed so that the zero is on the prolongation of CF, and ·1683 on the parallel to CF through P; for it is essential that this scale

1 Here O, P, F are reserved for the vertices of the triangle. In **4.14** Ex. 3 they were used for weights in grammes, and this use may be continued without confusion. Similarly c, p, f are used temporarily in **9.52311** for the lengths of the sides, but they are seldom required for this purpose.

whatever its position, correspond with *CP*, and that it be uniform.

Ex. 1. Verify by measurement on an accurate diagram drawn like figure 67 that *c*, *p*, *f* corresponding to *R.Q.*=0·824, $N/0_2$=0·090 are 22·8, 53·5, 23·7, as in figure 58.

Ex. 2. Show that the line joining a graduation x on *FC* (*i.e.*, no protein) and $100-x$ on *CP* is parallel to *PF*.

9.522. In figure 58 scales are marked for only two of the three constituents, *viz.*, O_C giving the percentage of oxygen used in the burning (*i.e.*, oxidation) of carbohydrate, and similarly O_P, or *p*. The percentage used for the oxidation of fat is got by the subtraction of the sum of these from 100. While *c* is marked along the horizontal side, which we have called, following the suggestion of **9.511**, *FC*, *p* is not marked along the inclined righthand side *CP*, and there is no scale on *PF*. *CP* is reserved for **two scales of** percentages of **calories** derived from protein and carbohydrate, and thus by the three parallel lines (rather unfortunately obscuring the fact that the only graduated line of the three is the side of the triangle) it is emphasised that the chief purpose of the diagram is to show the percentages in which heat is got from the different constituents of the food through combination with oxygen.

We have already seen in **4.14** Exs, 1, 2 how these heat scales for a varying combination of two constituents may be calculated and marked out. The non-uniform character of the heat scale here shown (which may of course be found also by measurement) is brought out by the way in which the parallels to *PF* through the calorie graduations cut *FC* an increasing, and then a diminishing distance to the left of the parallels through the corresponding *c* graduations. On *CP* the corresponding fact is masked because the parallels to *FC* are drawn through the graduations of N/O_2, and there is no simple regularity. (It should be noted carefully, however, that in figure 58 we have really only one heat scale—that in which no fat is being consumed, as given in **4.14** Ex. 2. Cf. **9.525** Ex.)

Ex. Prove *from the formulæ* for c (**9.5283** Ex. 1) and C' (**4.14** Ex. 1) that it is only for $R<1$ that corresponding to a given value of R the c graduation at a point is less than the C' graduation. Determine the corresponding property for the p and P' scales.

9.5231. The need of a really systematic method of arranging such diagrams is strikingly brought out by **two apparent defects** in figure 58. One is that, save in the original paper (J. B. C. **59** 57), the impression is given that by the same point are marked along the righthand sloping lines **independent** values of C' and P' which total 100.[1] In **9.5211** has been stated the corresponding property which *is* true for scales uniformly graduated, or possibly graduated by formulæ which have some suitable complementary character. But this is not the case for the non-uniform scales given by the formulæ of **4.14**.

From the original paper it becomes clear that there is actually only one scale here, and hence there is no need to show double graduations (cf. the note on Ex. 4, p. 76): the formula is that given in **4.14** Ex. 2, which may also be written $P'=\frac{448{\cdot}5p}{4{\cdot}485p+5{\cdot}047(100-p)}$, since the uniform scales, R between 1 and ·801, and p between 0 and 100, are equivalent. But it is difficult to see how consistency has been observed in the treatment of this scale: for convenience in getting C' for any point *parallels* have been drawn to PF, with the result that the scale has been transferred bodily to FC, where (**9.525**) the appropriate scale is that given in **4.14** Ex. 1; yet to get P' for any point it turns out (**9.525** Ex.) that this scale is connected by *unequally-tilted lines* with the appropriate heat scale on PF.[2]

Ex. Draw a trilinear diagram with c, p, f, scales uniform as before; but with the heat scales given by, say,

$$C'=100\ \frac{5(1/R-1/0{\cdot}7)}{5(1/R-1/0{\cdot}7)+4{\cdot}7(1-1/R)},\quad P'=100\ \frac{4{\cdot}5(R^2-1^2)}{4{\cdot}5(R^2-1^2)+5(R^2-{\cdot}8^2)},$$

where the values of C' and P' at F, C, P are the same as in figure 67. Test whether the join of graduations x on one scale and $100-x$ on the other is parallel to PF. Devise similarly some suitable formula for F'.

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1. This might be defended by saying that a point on CP of graduation x represents $F'=0$ and $C':P'::100-x:x$. The corresponding point on FC represents $P'=0$ and $C':F'::100-x:x$. Obviously, since the same percentage C' is represented in each case, the join of these points is parallel to PF. (The argument is even more impressive when presented in the "coincidence" form suited to figure 58.) But x is obtained from a different formula for each of these points.

2 In a similar diagram (though probably not drawn to scale) given by Du Bois on p. 42 (or J.B.C. **59** 45) there is no doubt as to the intention to show the heat scales as independent: otherwise the lines joining the heat graduations would have been parallel to FC. (The phrase "percentage of calories from protein" is ambiguous when it refers to a scale.)

(The point may be seen with less trouble if the heat scales are made of any arbitrary character, as different from the uniform scale as, *e.g.*, any of the stationary scales mentioned in **5.5** f. n.)

9.52311. In considering all the possible arrangements that could result from the nature of the formulæ which determine figure 58, it may be argued that the cyclic symmetry of the formulæ for the graduation of the heat scales may correspond to this parallel property : this is not so. But it is worth while analysing this supposition : it will be seen later (**9.5262**) what is the actual graphical effect of the cyclic order of the formulæ.

Taking the calorific equivalents of 1 litre of oxygen when carbohydrate, protein, fat alone are metabolised as respectively h_1, h_2, h_3 (ffg. 67,) we get the generalised expressions for C', P', F'

$$\frac{100\,h_1\,(R-\cdot7)}{h_1\,(R-\cdot7)+h_3(1-R)},\quad \frac{100\,h_2\,(1-R)}{h_2\,(1-R)+h_1\,(R-\cdot8)},\quad \frac{100\,h_3\,(\cdot8-R)}{h_3\,(\cdot8-R)+h_2\,(R-\cdot7)}$$

where R is not necessarily the same in each expression: it is chosen merely as a convenient uniform scale.[1] Expressing these in terms of lengths along the sides of the triangle we have for the graduations

$$C'=100\frac{h_1r_1}{h_1r_1+h_3(p-r_1)},\quad P'=100\,\frac{h_2\,r_2}{h_2\,r_2+h_1(f-r_2)},\quad F'=\frac{h_3\,r_3}{h_3r_3+h_2(c-r_3}$$

where r_1, r_2, r_3 are the actual lengths, measured along the sides one way round corresponding to the respective graduations ; and p, f, c are the lengths of the sides. Then, if, say, $C'+P'=100$, that the line joining these graduations be parallel to PF, we must have

$$\frac{r_1}{p-r_1}=\frac{f-r_2}{r_2}$$

$$\text{Thus } 100=C'+P'=100\left(\frac{1}{1+\frac{h_3}{h_1}\cdot\frac{r_2}{f-r_2}}+\frac{h_2\,r_2}{h_2\,r_2+h_1\,(f-r_2)}\right)$$

$$=100\left(\frac{h_1(f-r_2)}{h_1(f-r_2)+h_3\,r_2}+\frac{h_2\,r_2}{h_2r_2+h_1\,(f-r_2)}\right)$$

$$=100\,\frac{N^2+2N\,h_2\,r_2+h_2\,h_3\,r_2{}^2}{N^2+N(h_2+h_3)\,r_2+h_2\,h_3r_2{}^2},\qquad \text{where } N \text{ stands for } h_1\,(f-r_2).$$

This relation holds if $2h_2=h_2+h_3$, i.e., if $h_2=h_3$; which means that the same amount of heat would be obtained from either protein or fat metabolised by a given quantity of oxygen—a result we might have anticipated from our attempting to equalise the C' and P' scales. This also explains why parallelism (i.e., coincidence in figure 58) is so nearly attained : the difference h_2-h_3 is only about 5 per cent. of the value of either (cf. **9.524**.)

Ex. Apply this method to the formulæ for c, p, f, viz.,

$c=100\,(R-0{\cdot}707)\,/0{\cdot}293$ (Du Bois, *loc. cit.* p. 39), i.e., $100\,(R-0{\cdot}707)\,(1-0{\cdot}707)$ etc. : prove that in this case the join of graduations which total 100 *is* parallel to a side of the reference triangle.

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1 The first two of these expressions are the formulæ of **4.14** Ex. 1 and 2 respectively.

9.5232. The second defect is even more one of wrong impression: it appears as if **calorie graduations** of the non-uniform scale for P' on CF are **joined to** the numerically identical **graduations for** p carried from the uniform vertical scale for O_p to FP! One rebels against this obviously wrong suggestion: yet no hint is given in the figure as to the construction of the lines. From the original paper it can be found that the points on FP are determined by the formula for heat from the oxidation of only protein and fat, and that points representing the same percentages of heat derived from protein oxidised along with carbohydrate and with fat respectively are joined by broken lines. These lines, though no two of them are parallel, are used to fix the position of any point with respect to the Cal_p, *i.e.*, P', scale. Owing to the smallness of the tilt of any of these lines (**9.52311**), and the degree of accuracy attainable in the calculations (**4.14**, Ex. 3), the difference in the calorie reading through not taking parallels to FC is scarcely noticeable: but the principle is quite a new one and needs examination. Cf. **9.526.**

9.524. The diagram, fig. 59, which is also reproduced from Du Bois' work (p. 74), though on a reduced scale, is no argument for tilted lines! It is used to determine the **quantity** of metabolism from the same data which gave in figure 58 the *distribution* of the metabolism. The diagram is very simple: just as the vertices of the triangle are fitted into the *R.Q.* scale, so are they fitted into the scale for the heat that is derived from the oxidation of food (or body substance if the body is wasting) by a fixed quantity of oxygen: this heat varies according to the composition of the food, the calorie values from 1 litre of oxygen used in metabolising carbohydrate, fat and protein being 5·047, 4·686, 4·485 respectively.[1] Accordingly the construction of the figure is: graduate PC uniformly between 4·485 and 5·047—join the graduation 4·686 to F, and draw parallels to this line through the graduations on PC.

1. These numbers should be clearly distinguished from the multiples, 4, 4, 9 of **9.514**, Ex. 4, which convert *grammes* of the constituents into calories of heat derived from their consumption. Cf. also **4.14** Ex. 3.

Ex. Show that it is possible to draw the above figure with the oxygen calorie lines perpendicular to *CP*; how could the *R.Q.* rulings be then determined if they had not been already settled? Are other arrangements of the figure possible?

9.5241. "A few clinicians have neglected the protein metabolism in their calculations entirely, but this is done through gross ignorance.....If we derive 15 per cent. of the calories from protein, and this is a fair average, the calorific value of a liter of oxygen is 1 per cent. lower than" the figure from certain tables based on the oxidation of carbohydrate and fat only. (Du Bois, loc. cit. pp. 38, 70). These puzzling sentences find an easy explanation in figure 59. If a line is drawn parallel to *CF* through the graduation 15 on the Cal_p scale, (which here replaces O_p as the vertical scale) this represents metabolism in which 15 per cent. of the heat is derived from protein. Suppose the *R.Q.* to be 0·8; then, from the figure, the calorie equivalent for no protein would be 4·798, and for 15 per cent. protein-heat 4·752. The difference ·046 is less than 1 per cent of either of these measures.

Ex. One authority states that this difference arising from neglect of protein metabolism may amount to 3 per cent. in extreme cases. Show that the amount of heat derived from protein is, for such a case in which the *R.Q.* is 0·8, about 13 per cent. Prove also that when the discrepancy in the calculation is of this amount, a maximum amount of heat is being received from protein when no carbohydrate is being metabolised. Determine the amount of protein thus metabolised from a carefully drawn generalised diagram.

9.525. A GENERALISED REFERENCE TRIANGLE: Figures 58 and 59 can be combined in one, 60, so as to give almost perfect **cyclic symmetry** in the arrangement. This important simplification can be effected if the *R.Q.* scale is removed from *FC*, thus leaving the sides of the triangle free to carry the oxygen-combination and heat scales of the constituents, carbohydrate on *FC*, protein on *CP*, fat on *PF*. The *R.Q.* scale may be marked on a line close to *FC* but distinct from it. Parallels to the sides through convenient graduations of the uniform *c*, *p*, *f*, scales will cover the triangle with a network of similar triangles, though not equilateral as in figures 65 and 66. The oxygen-combination scales are thus on the inner sides of the boundary lines; very few of the graduations of these scales need be marked

with numbers. The heat scales can then be conveniently and freely marked on the outer sides of the lines. Then, as in **9.5211**, three lines drawn (best in imagination) *from* a point, determined by values of N/O_2 and the *R.Q.*, parallel to the sides in succession taken one way round, will meet the sides in points, each of which indicates the percentage of oxygen used by the constituent. What the relation of the point is to the non-uniform scales we cannot as yet say: this we shall examine in **9.526**.

The scale for the calorie values of oxygen which consumes any combination of constituents (**9.524**) can be marked outside the triangle on any convenient line (usually parallel to *CP*) crossing the parallel lines through these graduations.[1] Thus we have **nine scales** in the figure, two on each side of the triangle, and three outside; and these convey all the information that might have been given in figures 58 and 59.

Ex. In the Journal of Biological Chemistry, **59** 57, pairs of alternative formulæ in terms of P' and C' are given for graduating the heat-scales on the sides of the triangle *CPF*. (In each case one formula suffices, for the point (C', P') is on a given line.) The formulæ are

for *CP* (1) $N : O_2 = \frac{0{\cdot}8497\ P'}{0{\cdot}562\ P' + 448{\cdot}5}$ (2) $R.Q. = \frac{404{\cdot}3 + 0{\cdot}442\ C'}{504{\cdot}7 - 0{\cdot}562\ C'}$

for *PF* (3) $N : O_2 = \frac{0{\cdot}7889\ P'}{0{\cdot}201\ P' + 448{\cdot}5}$ (4) $R.Q. = \frac{375{\cdot}3 - 0{\cdot}582\ C'}{468{\cdot}6 - 0{\cdot}201\ C'}$

for *FC* (5) $R.Q. = \frac{468{\cdot}6 - 1{\cdot}118\ P'}{468{\cdot}6 + 0{\cdot}361\ P'}$ or (6) $c = \frac{468{\cdot}6\ C'}{504{\cdot}7 - 0{\cdot}361\ C'}$

where throughout $P' + C' = 100$. Examine these formulæ, *e.g.*, show that the denominators of (1) and (2) are equivalent, and that the ratio of the expressions for $N : O_2$ and $1 - R.Q.$ is the slope of *OP* when the scales for C' and P' are equal. With the help of $N : O_2 = {\cdot}001683p$, deduce the formulæ from the fundamental formulæ of **4.14** Exs. 1, 2 and a similar formula for protein and fat in terms of uniform-scale values—formulæ such as we begin with in **9.52311**. Is it right to use C' in (4) and P' in (5), though $c = 0$ along *PF* and $p = 0$ along *FC*? (Cf. the abstention from the use of $N : O_2$ for *FC*.) Is there any reason why the *R.Q.* should not be expressed in terms of P' in all cases? (Cf. **9.5231**) Test the formulæ on figure 58. Note how similarity of algebraic form in the columns would be obtained by interchanging (5) and (6); what significance has this?

(This investigation should be planned out with the whole class, and then the work divided among several sections.)

1 These lines, as well as the lines through the N/O_2 and the *R.Q.* graduations may be dotted or distinguished in some other clear way from the rulings for equal percentages of oxygen consumed by the constituents.

9.5251. This generalised figure may be **modified** in several ways to secure special conveniences. Suppose it were desirable to make the calorie-value-of-oxygen rulings perpendicular to *CP*, as well as the respiratory quotient rulings perpendicular to *FC*. This can be effected easily by drawing a perpendicular to *FC* through the point of *FC* which divides the interval between the graduations ·801 and 1 of the *R.Q.* scale in the ratio 4·686−4·485 : 5·047− 4·486, *i.e.*, 201 : 361 or approximately 1 : 1·8. A circle on *FC* as diameter will cut this perpendicular in a point which lies on the side *CP* of the required triangle—it is the point which has the calorie-value-for-oxygen graduation 4·686 The generalised figure, 60, is drawn with this special arrangement.

9.5252. The question at once arises: can the rulings for N/O_2 be made perpendicular to *PF*? This reveals an essential **lack of symmetry** in the figure; and this is easily explained by the fact that the N/O_2 scale is related to only two vertices, like the six scales on the sides, while the other two scales are related to three vertices. Another essential consideration for the N/O_2 rulings is that N/O_2 is quite independent of the *R.Q.*, and therefore its rulings must be parallel to the scale of the *R.Q.* We have thus a subsidiary complementary relation between the *R.Q.* and the N/O_2 rulings, the essential feature of which may be expressed by saying that *the* R.Q. *rulings must be parallel to the line joining the* R.Q. *graduation 0·801 and the vertex* P, *while the* N/O_2 *rulings must be parallel to* FC.

9.5253. If for any reason it is desirable that the oxygen-combination **scales** of the constituents be made all alike, *i.e.*, **with the same unit,** this may be effected by the consideration that the scales marked along the sides may be laid quite as well along any line through the vertex at either extremity of the side to the side opposite this vertex. Then, taking a length equal to at least the longest perpendicular from a vertex to the opposite side, say *CH* (fig. 68), we can mark on it one scale *c*, and on *KP* and *LF* scales *p* and *f*, where *PK* and *FL* are lines through the respective vertices equal to *CH*. This, however, has no

significance for the heat scales. The rulings for the weight scales in this case are not affected: the lines *CH*, *PK*, *FL* would merely have to be specially marked, say, in colour.

9.52531. This gives a means of measuring off directly from their respective scales the relative weights of the constituent substances. In the metabolism of protein the ratio of the weights of the protein and the oxygen involved in the process is ·725 : 1; for carbohydrate the corresponding ratio is ·841 : 1; for fat it varies—call it r in a particular case. Accordingly, if in figure 68 *K*, *H*, *L* are placed so that $PK : CH : FL :: ·725 : ·841 : r$; then the lengths on these lines, which represent percentages of oxygen used will, when transferred to a suitable standard scale (for 100 gms., say), represent the relative weights of the constituents metabolised.

9.526. But there is a more direct way of dealing with the relative quantities of constituents metabolised; and in connection with it we shall consider the relation of **points within the triangle** to non-uniform scales on the sides, rather then in connection with the nearly uniform heat-scales. It has been determined experimentally that the number of grammes of carbohydrate, protein and fat which can be consumed by a litre of oxygen are **1·21, 1·03, 0·50,** respectively. With these values formulæ like those given in **4·14**, are constructed to determine what weights of pairs of these substances may be combined in oxidation by 1 litre of oxygen. These are the **percentage weights** of **9·5211**, γ the percentage weight of carbohydrate when carbohydrate and fat are burned; and so for π and ϕ: thus

$$\gamma = \frac{121\,(R-·707)}{1·21\,(R-·707)+·5(1-R)} = \frac{121c}{1·21c+·5(100-c)}, \text{ etc.}$$

Ex. 1. Combine the above values with those given in **9.524** of the calorie values of substances combining with 1 litre of oxygen, to find the calorie value of 1 gm. of each of carbohydrate, protein, fat. Cf. the result got in **9·514** Ex. 4.

Ex. 2. In the Journ. Biol. Chem. **59** 58 the following formulæ are given for graduating the sides of the triangle *CPF* with scales which show the percentage weights of the substances metabolised.

$$(1)\quad N:O_2 = \frac{0{\cdot}204\,\pi}{103+0{\cdot}18\pi} \qquad (2)\quad R.Q. = \frac{97+0{\cdot}06\gamma}{121-0{\cdot}18\gamma}$$

$$(3)\quad N:O_2 = \frac{0{\cdot}084\,\pi}{103-0{\cdot}53\pi} \qquad (4)\quad R.Q. = \frac{40+0{\cdot}33\gamma}{50+0{\cdot}53\gamma}$$

$$(5)\quad R.Q. = \frac{50+0{\cdot}36\pi}{50+0{\cdot}71\pi} \quad \text{or} \quad (6)\quad c = \frac{50\gamma}{121-0\,71\gamma}$$

Examine these as the heat formulæ were examined in **9·525**, Ex.

Ex. 3. Fit to the reference triangle rulings showing the *absolute weight* of foodstuff per litre of oxygen, just as in **9.524** rulings to show the calorific value of a litre of oxygen were fitted. (The essential line for this is shown in figure 60.)

9.5261. To consider what significance a point within the triangle has with respect to non-uniform scales, we might define its position by means of the scales for N/O_2 and the *R.Q.*, which are convenient from the experimental side. But it is simpler to use two of the uniform scales along the sides of the triangle, say, c and p, to give oblique co-ordinates (**5.4**)—note that the N/O_2 and the *R.Q.* scales might have been oblique (**9.5211**). Then our problem is to find what relative weights of the components consume oxygen in the proportions $c : p : 100-c-p$.

The number of grammes of the substances that are combined in the consumption of 100 litres of oxygen are respectively $1{\cdot}21\,c$, $1{\cdot}03\,p$, $\cdot5(100-c-p)$. Therefore immediately, for any mixture of the three,

$$\gamma = \frac{100\times 1{\cdot}21c}{1{\cdot}21c+1{\cdot}03p+{\cdot}5(100-c-p)} = \frac{121c}{50+{\cdot}71c+{\cdot}53p}, \quad \pi = \frac{103p}{50+{\cdot}71c+{\cdot}53p},$$

$$\phi = \frac{50(100-c-p)}{50+{\cdot}71c+{\cdot}53p}.$$

Substitution of the values of c and p in these expressions gives us a first answer to our question.

9.5262. It is well, however, to press it further and ask about the lines in the figure along which these percentage weights are constant. If we take, say, γ as constant, the corresponding formula above becomes a linear relation between c and p, which, we have learned, represents, even in oblique co-ordinates, a straight line. Working this out, we get $\cdot53\gamma p=(121-\cdot71\gamma)c-50\gamma$, or $p=(228/\gamma-1{\cdot}34)\,c-94{\cdot}5$. If in this we substitute for γ the values 10, 20,.....90, we

get **a series of straight lines** along each of which the proportion by weight of carbohydrate consumed is constant. Similarly for the other constituents.

To construct the figure the simplest procedure is to mark the **scales along the sides,** as in figure 60, and then join by straight lines graduations which total 100. Graphical confirmation of the corrrectness of the above reasoning will be found in the fact that lines for which the total percentage weights of the three constituents taken in order is 100 intersect by threes: thus a triangle, ruled regularly as just stated, becomes covered with **a network of triangles,** though not of constant shape. In figure 60 this network is not shown, as there the more fundamental parallel rulings, between which it is easier to judge intervals, had to be left clear. But the three lines drawn, as suggested in **9.5211** for parallel rulings, from the point for which $\gamma = 40$, $\pi = 40$, $\phi = 20$, show clearly how the weight-rulings would appear.

In *graduating the scales* it is natural to express each weight in terms of the corresponding percentage of oxygen used; thus for π we have

$$\pi = \frac{103p}{1{\cdot}03p + 1{\cdot}21(100-p)} \therefore p = \frac{121\pi}{103 + {\cdot}18\pi}$$: note that the correctness of the numerical coefficients may be checked by putting $\pi = 100$. Then the calculation would be as shown here. The denominators increase by equal steps, and may be inserted first, as indicated. The checks are shown in brackets.

π	20	40	60	80	(100)
p	$\frac{2420}{106{\cdot}6}$	$\overline{110{\cdot}2}$	$\overline{113{\cdot}8}$	$\overline{117{\cdot}4}$	$\left(\frac{1210}{121}\right)$
	2·27	43·8	63·7	82·4,	

Ex. 1 Verify the other two percentage-weight scales in figure 60, and join by pencil lines complementary graduations. Complete the percentage-heat scales, and similarly draw the triangular network in red lines.

Ex. 2 Explain the fallacy in the following: "To find the significance for non-uniform scales of points within the triangle, it suffices to draw parallels to the sides, as suggested in **9.5211,** and take the ratios of the three readings thus obtained, though their sum is not 100. We have seen (cf. **9.5253**) that the direction of the line through the vertex on which a scale is marked makes no difference, and therefore there is no change in the significance of a point if it is transferred to any position on the line through it parallel to that on which the extremity of the scale moves. It may also be urged that the use of lines parallel to the sides has reference primarily to the uniform scales for c, p, f; and the relation of these uniform scales to the non-uniform scales for relative heat and weight holds good for each pair of the constituents. Hence the above simple procedure suffices."

9.527. There is no intention in this book to suggest anything contrary to the view that these studies of the mathematical aspects of diagrams used in medicine are quite inadequate to give an understanding of the nature of the medical facts to which they refer. Yet it would scarcely be right to leave the subject in the jumbled state to which mathematical simplicity has reduced it: and so an outline is given here of the procedure which is followed in using these diagrams and in making the associated calculations described in **4.14**, etc.

The starting point is in **9.521** where, as there pointed out, two clinical measures give a point from the position of which something can be inferred as to the nature and the amount of the metabolism of the individual considered. Of these two measures much the more important is the *R.Q.* This is obtained by the person examined[1] breathing into an apparatus which measures the total volume of air expired, and the composition of that air is found by analysis. The latter gives the *R.Q.*, and hence from figure 59 the actual calorie value of a litre of the oxygen used in the metabolism: this, combined with the former, the total volume of oxygen used, gives the total number of calories per hour produced in this metabolism.

This figure is then divided by the surface area of the body, as determined by **5.3** Ex. 4, to give the **basal metabolic rate,** which is compared with the normal standard shown on the lefthand scale of figure 1, just as temperature during fever is compared with normal temperature. This normal standard varies with age and with sex, but comparison with it is not rendered thereby less valuable: even "normal" temperature is not an absolute constant—the doctor makes allowances for individual peculiarities. And so the basal metabolic rate becomes in diseases such as those in which there is abnormal thyroid activity (for which formerly treatment was almost by intuition only[2]) as important an indication of the condition of the patient—"a measure of toxicity"—as temperature is in the case of pneumonia.

9.5271. The other diagrams and calculations relate to adjustment of **diet** in normal or abnormal cases. For this it is necessary to know the total requirements of the body for a day. This can be determined for either health or disease from the nomogram, fig. 1; cf. also **5.241** Ex. 2. How these calories may be supplied can be determined from **5.22** Ex. 4 or **9.63** in the case of diabetes. In health the body can consume the three constituents in almost any proportions: cf. **9.514** Ex. 6. When either the calorie values or the weights of the three constituents have been determined, the methods of **9.51** may be used to help to determine the most suitable foods to be given and their weights: a person's activity has to be considered as well as the above experimental data, which refer only to minimum requirements. Note specially **9.63** Exs. 1, 2, 3.

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1. In the Journ. of Biol. Chem. **59** 82 a modification of the conditions under which a patient must be examined, as stated in **7.53** f.n., is shown to be permissible.

2. With no definite guide beyond impressions as to the effect of treatment, the lot of the doctor who had to decide in a case of rapidly wasting disease between the need and effectiveness of drugs, rest, X-ray treatment, or surgery, could not have been a very happy one.

9.5281. It does concern us as mathematicians to see what meaning the **space outside the triangle** of reference may have; this is illustrated by figure 72 (Du Bois, *loc. cit.*, p. 44). The region within the main triangle to the right of *PF* represents conditions under which carbohydrate is being consumed; the region to the left, conditions under which it

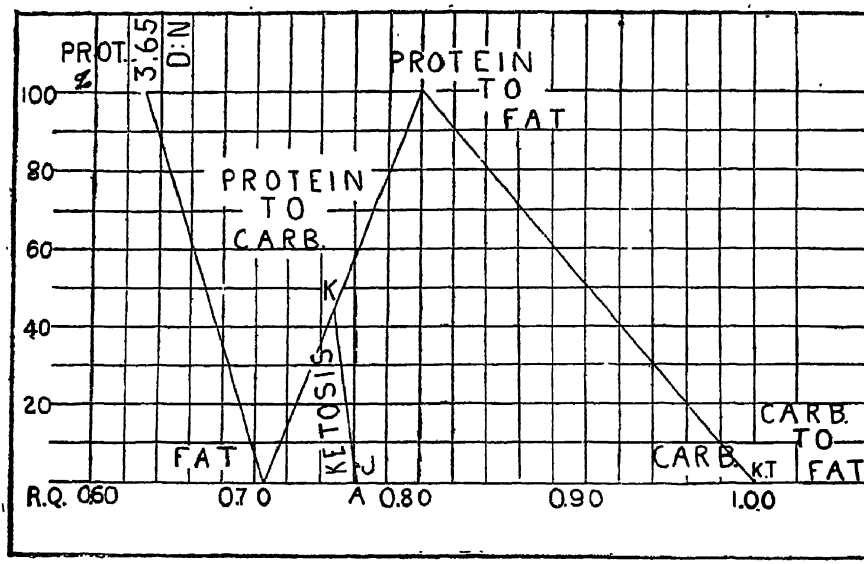

Fig. 72. "Zones of Metabolism."

is being produced. Similarly the production of fat from either protein or carbohydrate is represented to the right of *CP*. In mathematics these reversed processes would be represented by negative coordinates—the distances from the sides, in whatever directions these be measured. Cf. **1.5.**

These points beyond the bounds do correspond to observed conditions, *e.g.*, the *R.Q.* of geese fed liberally with grain was found to rise to 1·33: this means that they could not breathe in enough oxygen to use all the protein and carbohydrate with which they were stuffed, and had to store part of it as fat. But it is not necessary that the *R.Q.* be greater than 1 that protein should be changed into fat.

The letters placed on the diagram refer to extreme cases: the lowest observed R.Q. was 0·632: the 3·65, marked in the figure, was in this case what is called the

"dextrose-nitrogen" ratio. (Dextrose is the G of **4.14** Ex. 3.) This is the ratio of sugar (59·41 gms. per 100 gms. protein in this case) to nitrogen (16·28 gms. as in **9.521** f.n.) found in the urine of a dog in a diabetic condition, which was being given no carbohydrate (Du Bois, *op. cit.*, p. 205). Both these substances must come from the same protein; hence the extent to which protein is being converted into carbohydrate is determined. The value of this ratio, 3·65 : 1, has become "the foundation of all modern calculations in diabetes." "It is only in the lower portion of this (second) triangle that we find patients with diabetes, since they seldom derive more than 35 per cent. of their calories from protein." (J. B. C. **59** 48).

The line KA represents "the threshold of ketosis," and is said to correspond to figure 74. The equation to the line should be deriveable from the fundamental formula given in **9.63**, by substituting $100-P-C$ for F. This gives

$$100=3C+1{\cdot}55\,P,$$

whence is got a value for the intersection with FC which agrees well with figure 72; the agreement for the intersection with FP is not so good.

Ex. By what lines are $F=2C$, $F=3C$ represented in the triangular diagram?

9.5232. In **9.521** we allowed the impression made by the figures to serve as a proof of the possibility of constructing, as in **9.51**, a **constant sum** for the lengths of the graduated perpendiculars to the sides. This is obviously not the case in a scalene triangle for perpendiculars measured in the same unit. To find what is involved in this for the choice of scale unit, let us suppose a point which represents the measures, c, p, f, in a certain metabolic condition. Let μ_1, μ_2, μ_3 be the respective units of the uniform scales along the perpendiculars. Then, by geometry,

$$c\mu_1.PF+p\mu_2.FC+f\mu_3.CP=\text{twice the constant area.}$$

Accordingly, put

$$\mu_1 : \mu_2 : \mu_3 :: FC.\ CP : CP.\ PF : PF.FC;\ \text{then}$$
$$c+p+f=\text{a constant,}$$

which may be taken as 100. Note that **9.51** is a particular case of this.

9.5283. The use of non-uniform scales along the sides of the triangle of reference suggests the possibility of generalising the trilinear system of reference in a way similar to the generalisation of the Cartesian system in **9.3**, **9.4** and

7.55. We can easily imagine how the straight rulings for respiratory quotients, or those for calorific value of oxygen may be displaced parallel to themselves or tilted, when the uniform scales along the sides are for calories from, not oxygen used by, the component substances; or when the proportions of the triangle are altered. But there seems no reason why more complete modifications may not be effected in such figures by the use of suitable scales, or by altering the shape of the triangle. Thus, if the density and strength curves in figures 65 and 66 are ellipses, it may be possible to convert them into circles.[1]

Ex. 1. Du Bois (p. 39) gives the obvious formula (referred to in **9.52311** Ex.) $c = 100(R - \cdot 707)/\cdot 293$ By substitution of R in terms of O', from the equation of **4.14** Ex. 1, in the equation for c, obtain an expression for c in terms of O'.

(This will enable a scale for O_c to be marked when the Cal_c scale is made uniform; etc.)

Ex. 2 (*For doctors.*) Consider any advantage that may result from modifying the diagram; e.g., by making the heat scales uniform; by making the triangle simpler, either isosceles or equilateral. (In the first of these cases the *R.Q.* and the nitrogen-excretion scales would be non-uniform; but once these are graduated there is no difficulty due to this in using them.)

Ex. 3. Devise a nomogram in which can be represented all that is shown by the triangular diagram, figure 60.

9.6. Straight-line graphs may occur with scales of reference other than uniform or logarithmic, and may be used to represent quantities inter-related in many ways. We conclude our study of graphs with a few examples which illustrate this statement.

9.61. BALLOON GRAPHS. In meteorology knowledge of the upper air has become of great importance. One source of information is exploration by means of small balloons which carry recording instruments, or other means of securing records, to great heights and distances. It is of great advantage to be able to determine quickly the position of

1 For the whole idea compare the very elementary treatment of Homogeneous Strain in Lamb's "Infinitesimal Calculus", Art. 131; also "Nature" **114** 9, "The use of 'Shear' in Geometry".

the balloon at regular intervals while it is in sight.[1] The ordinary methods of trigonometry are so cumbersome as to be useless in this case. The distance and height of the balloon are obtained with sufficient accuracy by observing the **apparent length** L **of a tail** of known length n metres attached to the balloon, and also the elevation E of the

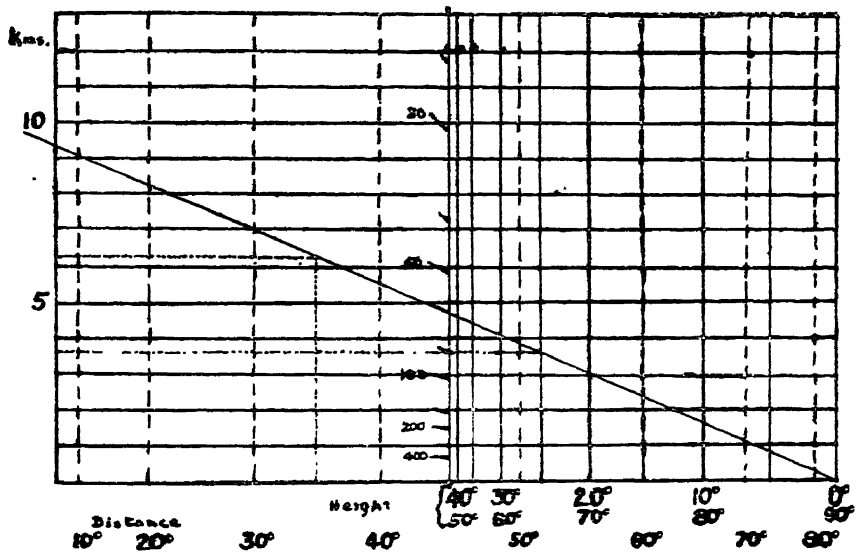

Fig. 73. Height and Distance of a Balloon.

(In error the height for an elevation of 25° has been marked instead of for 35°: but it is easy to read off, without any guiding line, the height 4·4 kms.)

balloon. The latter is obtained in the usual way from a theodolite (which also gives the direction of the balloon), the former is measured on a special scale fixed inside the theodolite-telescope in the field of view. On the supposition that the tail hangs vertically, formulæ for height h and

1 "In India two kinds of balloons are used in meteorological work (and also for help in aviation); one, instrument-balloons carrying barographs, thermographs etc.; and the other, pilot balloons which are simply small hydrogen-filled balloons let off in order to find the direction and velocity of wind at different heights. In the case of the latter, the essential thing of course is to find the position of the balloon at different times after release".

distance d can be obtained easily with the help of **1.9**, and the standard formulæ in trigonometry,

$$\sin 2A = 2\sin A \cos A,\quad \cos 2A = 2\cos^2 A - 1\ ;\ \text{thus (fig. 69)}$$

$$OB = OB' = (n \cos E)/L \ \therefore\ h = OB \sin E = (n/L)\ \tfrac{1}{2}\sin 2E,$$

$$\text{and } d = OB \cos E = (n/L)\ \cos^2 E,$$

equations which may both be taken to represent straight lines, through the origin and of slope n/L, the scales of abscissæ being determined by functions of the elevation.

The values of h and d can be read with great ease from a diagram such as figure 72 : [1] a working diagram may have a scale about five times as great as that shown. The **origin** of coordinates is taken **on the right**, so that a string fixed there may be moved over the figure with the left hand while the right records the results. The slope of this line is determined by n/L, which is marked as a scale of reciprocals on the central vertical line—n being a constant, 25m, or 100m., as the case may be. The elevation is marked on the horizontal axis in two scales, one for $\frac{1}{2}\sin 2E$ to give the height (continuous ordinates), the other for $\cos^2 E$ [i.e., $\frac{1}{2}(\cos 2E + 1)$] to give the distance corresponding (broken ordinates). Both distance and height can then be read off on the same uniform scale of ordinates, conveniently placed, for the same units and axes are used in both cases.[2]

The scales here are shown right up to 90°, but in practice they are rarely needed beyond 45°. The diagram can be adapted to similar problems which occur when there are two observing stations in use : cf. J. H. Field, Memoirs of the Indian Meteorological Department XXIV v, Plate 5.

Ex. 1. Test the diagram by commonsense considerations, such as, that a height and a distance must be equal for $E = 45°$

Ex. 2. Sketch the arrangement of alignment nomograms which represent the above formulæ for h and d. Which diagram is the more advantageous ?

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1 Adapted from H. Jameson. Bulletins of the Colombo Observatory, I 6.

2 Note the similarity and the symmetry of the scales for $\frac{1}{2}\sin 2E$ and $\frac{1}{2}\cos 2E + \frac{1}{2}$. This is elucidated in figure 70, whence it can be seen that, e.g., the scale for $\frac{1}{2}\sin 2E$ from 45° to 90° is identical with that for $\cos^2 E$ from 0° to 45°, i.e graduations 45°, 55°, 65°,.....in one scale correspond exactly with those for 0° 10°, 20°, 30°.....in the other.

9.62. STRENGTH OF SLABS. Figure 71 is of a type you should recognise readily: it is very common in books on engineering.[1] The formula represented is $B=\frac{5}{6}wl^2$ which we would naturally represent by a nomogram as in **5.3**. But here it is represented by straight lines radiating from the origin, obtained by taking successively different values of l as marked on the lines. B represents what is called the bending moment per foot width of a cement slab, w the load (either stationary or moving) in lbs. per square foot, and l the span—the distance between the supports of the slab; this last is taken as constant to determine the slope of each line.

Ex. 1. For a "live" load of 224, a "dead" load of 75, and a span of 10 feet, show from the graph that the value of B is 29,000.

Ex. 2. Draw a chart to show bending moment in a slab where the concrete is such that the suitable formula is $10B=wl^2$.

Ex. 3. Taking successive values of the *height-weight index*, W/H, between ·42 and ·92, draw a chart to represent the area of the body, $A=1{\cdot}237\ (2W/H-4H\sqrt{W/H})$, where W is measured in kilograms, H in centimetres. (Du Bois, *op. cit.* 147.)

9.63. DIABETIC DIET. Very special care has to be taken by doctors in arranging for the food eaten by patients suffering from diabetes, a disease which results from a weakening of the **carbohydrate-burning** mechanism. Figure 74, taken from Du Bois' "Metabolism", p. 233,[2] is a diagram that has been used for the purpose of calculating the weight of carbohydrate, protein and fat which may be given to such patients: the nomogram of **5.22** Ex. 3 serves the same purpose, but the theory of treatment on which it is based is different; the formulæ are given there. The theory here is that suggested in **4.14** Ex. 3, where it was stated that the ratio FA/G should be less than 1·5; but with the introduction of insulin (which increases the oxidation of carbohydrate) it has been possible to increase this ratio with beneficial results.

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1 Adapted from the Public Works Department Handbook I, 387.

2 Originally from the John Hopkins Hospital Bulletin, **33** 128, from which the examples are also taken.

$$\text{If } 1{\cdot}5=\frac{FA}{G}=\frac{0{\cdot}46P+0{\cdot}9F}{C+0{\cdot}58P+0{\cdot}1F}$$

$$\text{then } F(\cdot 15-0{\cdot}9)=P(0{\cdot}46-0{\cdot}87)-1{\cdot}5C$$

$$\text{or } F=2C+\tfrac{41}{75}P = 2C+0{\cdot}5466P \doteqdot 2C+\tfrac{1}{2}P,$$

a relationship which all points on the graph have. Accordingly the graduations for F/C on the left of the diagram begin at 2.

(i) The equations to the lines through the origin are easily deduced. For, say, 10% of calories from protein we *eliminate* F thus :

$$\frac{4P}{4C+9{\cdot}3F+4P}=\frac{1}{10}, \text{ i.e., } 36P=4C+9{\cdot}3F=4C+18{\cdot}6C+5{\cdot}07P$$

$$\text{or } 30{\cdot}93P=22{\cdot}6C, \text{ i.e., } P={\cdot}731C,$$

which is the equation to a straight line.

(ii) By *eliminating* P between this and the fundamental equation we get the ratio $F:C$ which must hold for this particular case:

$$\frac{F}{C}=2+0{\cdot}5466\,\frac{P}{C}=2+0{\cdot}5466\times{\cdot}731=2{\cdot}399.$$

(iii) The position of the line AB, on which the ratio F/C is marked so as to coincide with the P graduations, can be determined in this case, or in any other, as follows: suppose $F/C=3$ corresponds with $P=50$, then

$$3=2+0{\cdot}5466\ P/C \quad \therefore C=50\times{\cdot}5466=27{\cdot}3.$$

The calculation $P=\frac{1}{9{\cdot}8}$ (Total cal.$-\text{Cal}_p-\text{Cal}_c$) is avoided by this graphical device.

(iv) To get the lines showing total calories we consider how many calories a point (C, P) represents. The corresponding ratio F/C is represented on AB by a length $P.\ 27{\cdot}3/C.$ and so for the correspondence selected for the scales in this figure, viz., 2 in F/C corresponds to 100 gms. P,

$$F/C=2+\tfrac{2}{110}\ P.\ 27.3/C \quad \therefore F=2C+{\cdot}546P,$$

which is just the fundamental formula.

$\therefore$ the total calories, $M = 4(C+P) + 9{\cdot}3(2C + {\cdot}5466P)$

$= 22{\cdot}6C + 9{\cdot}08P$

Thus, for M=1000, the intercepts on the axes are 44·2, 110. Also the equal calorie lines have the constant slope $-22{\cdot}6/9{\cdot}08$.

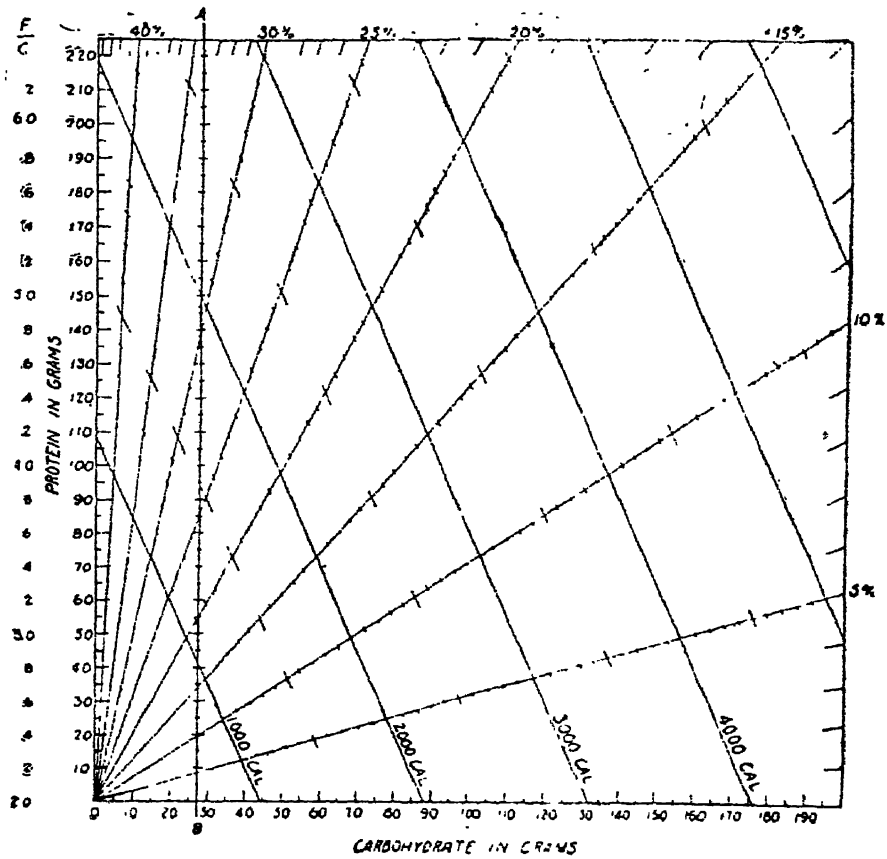

Fig. 74. "To calculate a diet formula, select the diagonal line representing the number of calories required; choose the radial line corresponding to the percentage of the total calories to be furnished by protein; from the intersection of these two lines read off the amounts of protein and carbohydrate on the axes; the intersection of the line AB with the radial percentage line determines the factor F/C. This factor multiplied by the number of grams of carbohydrate equals the grams of fat required. (Hannon and McCann,)"

Ex. Get a general equation for the line through the origin showing $Cal_p = x°/_o$. Obtain also the position of AB from a value y of F/C.—*i.e.* generalise (iii)

The purpose to be kept in view in the use of this diagram is said to be to keep **protein** metabolism **as low as possible** (cf Ex. 3 below) to give "the minimal amount of carbohydrate and maximal amount of fat that will avoid ketosis". Formerly an arbitrary allowance of 1 gm. per kgm. of body-weight had commonly been given : with this treatment it became possible to reduce protein to ·66 gms. per kgm.

Note that the diagram shows percentages of heat derived from protein up to what actually occurs in the case of Eskimos (**9.514** Ex. 6).

Ex. 1. Show that for $M = 2000$ cals., $Cal_p = 10°/_o$, the proper weights are $P = 49$, $C = 68$, $F = 162·5$; and that these give a total of 1920 cals. and a maximum glucose available 113. (**4.14** Ex. 3)

Ex. 2. Verify on figs. 1 and 74 the following statement : A man 30 years old, 170 cms. in height, weighing 60 kgms. requires at least 1612 cals./day. After a period of observation on a diet furnishing $5°/_o$ Cal_p, it is found that his protein metabolism as shown by the urinary nitrogen excretion, has reached a minimum of 50 gms. per diem. Reference to the chart shows that this constitutes $12·6°/_o$ of his total metabolism. If a diet is to be given which will just cover this minimal protein, requirement (50gms. P), the quantity of C which is prescribed will be 51gms., and of fat $2·53.51 = 129$ gms.

Ex. 3. The following quotation may, with the help of a large dictionary, be illuminating : "It will be found interesting to make from the graph the following calculations of two diets each furnishing 2000 cals. In one case the P forms $40°/_o$ of the total energy value. In the other case P forms only $10°/_o$ of the total energy. It will be seen that a patient, whose tolerance might be just sufficient to permit the taking of 2000 cals. with the smaller amount of P, would probably develop hyperglycemia or glycosuria if the same number of calories were given with the higher percentage of protein. The diet containing less protein permits the use of much greater amounts of free carbohydrate."

P	F	C	Available Glucose
195	125	9	134·6
49	165	68	112·9

And Satan stood up against Israel, and provoked David to number Israel.

1 Chronicles xxi 1.

Having eyes, see ye not? and having ears, hear ye not? and do you not remember?

When I brake the five loaves among five thousand, how many baskets full of fragments took ye up? They say unto him, Twelve.

And when the seven among four thousand, how many baskets full of fragments took ye up? And they said, Seven.

And he said unto them, How is it that you do not understand?

Mark's Gospel viii 18—21.

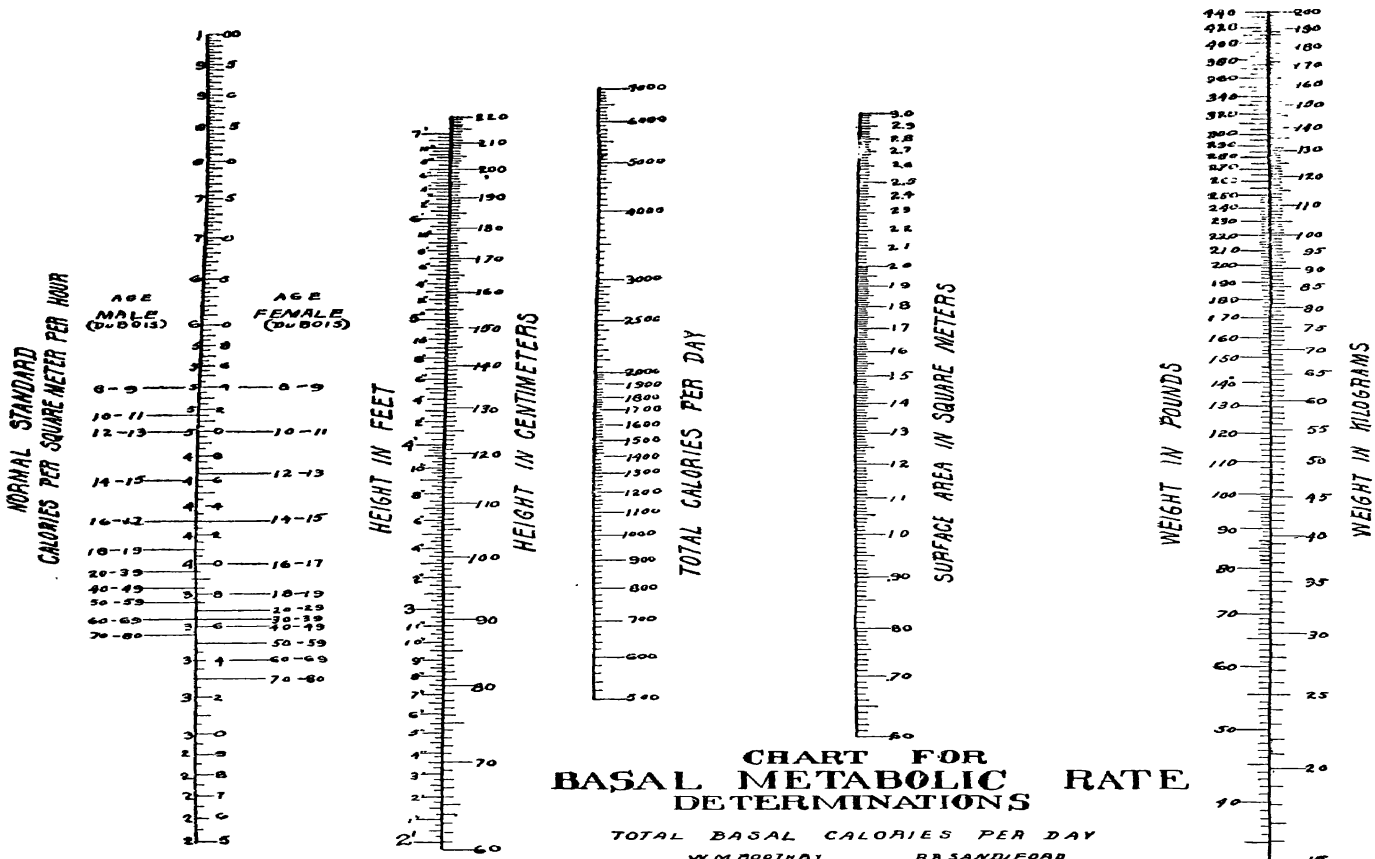

Figure 1. " Boothby and Sandiford's chart for quick and accurate calculation of normal basal food calory requirement. The weight is located on the weight scale and the height on the height scale with pins (needles set in wooden handles). A straight edge connecting these two points crosses the surface area scale at the patient's surface area. The normal standard is located on the lefthand scale. The line joining this point with surface area crosses the calory scale at the basal twenty-four-hour calory requirement." (Du Bois.) 5.31.